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Scalar multidimensional conservation laws IBVP in noncylindrical Lipschitz domains

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Abstract

We study the initial-boundary-value problems for multidimensional scalar conservation laws in noncylindrical domains with Lipschitz boundary. We show the existence-uniqueness of this problem for initial-boundary data in L^∞ and the flux-function in the class C^1. In fact, first considering smooth boundary, we obtain the L^1-contraction property, discuss the existence problem and prove it by the Young measures theory. In the end we show how to pass the existence-uniqueness results on to some domains with Lipschitz boundary.

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1. Introduction

In this paper we study the initial-boundary-value problems for multidimensional scalar conservation laws in noncylindrical domains with Lipschitz boundary. We prove the existence-uniqueness of this problem for the flux-function in the class C^1 and the initial-boundary data in L^∞, considering regularizable Lipschitz domains (see Definition 4.1). The L^1-contraction property, which establish stability, is obtained for domains with smooth boundary, at least C^2.

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It is well known that the initial-boundary-value problem, also called IBVP, generally is not well posed; an interesting description can be seen in Serre [17, vol. 2]. The important question is how we should consider the boundary condition. The qfirst work in this way was Bardos et al. [1], which extend to bounded domains the fundamental paper by Kruzkov [11] for Cauchy problems. They established that the boundary condition is given by an inequality to be verified for almost everywhere point at the boundary. This is possible since they consider data in BV, so the notion of trace at the boundary exists in a strong sense. Moreover, they have considered cylindrical domains with piecewise smooth boundary. The question on how the boundary condition should be assumed, was further studied by DuBois and LeFloch [5]. They observed the equivalence from their notion and that one given by Bardos et al. [1] in the scalar case. It was Otto [15] who proved the well-posedness of the IBVP for L^∞ data. From the notion of boundary entropy pairs he introduces a weak formulation in which sense we shall consider the boundary condition. It is important to notice that Otto has proven the L^1-contraction property for the flux-function in C^1 and existence for the flux-function in C^2, considering cylindrical domains with smooth boundary. Again for these type domains, Kondo and LeFloch [10] uses the notion of boundary condition proposed in Joseph and LeFloch [9], to prove existence-uniqueness and compactness results in a class of entropy measure-value solutions to the IBVP.

The notion of normal traces for L^∞ or even L^p fields is fundamental to ensure a precise notion for the boundary condition. This question is addressed by Chen and Frid [2,3], who introduced the notion of Divergence-Measure Fields, denoted by $\mathcal{DM}$, of L^∞ or L^p fields whose divergence is a Radon measure and, they generalized for these fields the Gauss–Green theorem. Finally, we mention the strong trace result obtained by Vasseur [21], with the aid of normal traces, generalized Gauss–Green theorem, cited above, and the kinetic formulation introduced by Lions et al. [12]. Vasseur shows this result, with a nondegeneracy condition on the flux-function, moreover in the class C^3. Hence, the weak boundary condition introduced by Otto [15] is equivalent to that one given by Bardos et al. [1]. Notice that this equivalence is still open for a general flux-function.

Let Q_T be an open subset of $\mathbb{R}^{n+1}$, whose points are denoted by $(t,x)\in\mathbb{R}\times\mathbb{R}^n$. We will denote by Γ_T the lateral boundary of Q_T and by Ω the set $\overline{Q}_T\cap\{t=0\}\neq\emptyset$. We are concerned with the following IBVP:

Find $u: Q_T\to\mathbb{R}$, satisfying

$$u_t+\mathrm{div}_x f(u)=0 \quad \text{in } Q_T, \tag{1.1}$$

$$u=u_0 \quad \text{in } \Omega, \tag{1.2}$$

$$u=u_b \quad \text{on } \Gamma_T, \tag{1.3}$$

where $f \in C^1(\mathbb{R};\mathbb{R}^n)$ is a given map called flux-function. We assume that the initial-boundary data satisfies

$$u_0 \in L^\infty(\Omega;\mathscr{L}^n) \quad \text{and} \quad u_b \in L^\infty(\Gamma_T;\mathscr{H}^n), \tag{1.4}$$

where $\mathscr{H}^s$ denotes the s-dimensional Hausdorff measure and $\mathscr{L}^n$ the n-dimensional Lebesgue measure (usually not written). The domain Q_T is noncylindrical and Γ_T is a Lipschitz n-variety, that are considered in the following form. Let $\mathscr{L} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a bi-Lipschitz map such that

$$\mathscr{L}(t,y) \coloneqq (t, x(t,y)) \quad \text{for all } (t,y) \in \mathbb{R}^{n+1},$$

where $x : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a certain Lipschitz function satisfying $x(t,y) = y$ for $t \leqslant 0$. By Rademacher's Theorem $\mathscr{L}(t,y)$ is differentiable $\mathscr{L}^{n+1}$ a.e., and therefore $D\mathscr{L}(t,y)$ exists and can be regarded as a linear mapping from $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+1}$ for $\mathscr{L}^{n+1}$ a.e. $(t,y) \in \mathbb{R} \times \mathbb{R}^n$. Since $\mathscr{L}$ is a bi-Lipschitz map, the Jacobian of $\mathscr{L}$ is positive, that is,

$$J\mathscr{L}(t,y) \equiv [\![D\mathscr{L}(t,y)]\!] > 0 \quad (\mathscr{L}^{n+1} \text{ a.e.}).$$

Let $\Omega \subset \mathbb{R}^n$ be an open set with regularly deformable Lipschitz boundary $\partial\Omega$, see Chen and Frid [2]. Set

$$Q \coloneqq \mathscr{L}(\mathbb{R} \times \Omega), \quad \Gamma \coloneqq \mathscr{L}(\mathbb{R} \times \partial\Omega),$$

and for some $T \in \mathbb{R}$, $T > 0$

$$Q_T \coloneqq \mathscr{L}((0,T) \times \Omega), \quad \Gamma_T \coloneqq \mathscr{L}((0,T) \times \partial\Omega).$$

Moreover, we can write

$$Q_T = \bigcup_{0<t<T} \Omega_t \times \{t\}, \quad \Gamma_T = \bigcup_{0<t<T} \partial\Omega_t \times \{t\}.$$

Now, we assume that $\Theta : \partial\Omega \times [0,1] \to \overline{\Omega}$ is a regular Lipschitz deformation for $\partial\Omega$ and fix a standard regular Lipschitz deformation

$$\psi : \Gamma \times [0,1] \to Q$$

for Γ, given by

$$\psi(r,s) \coloneqq \mathscr{L}(\pi_1 \circ \mathscr{L}^{-1}(r), \Theta(\pi_2 \circ \mathscr{L}^{-1}(r), s)),$$

where $\pi_1 : \mathbb{R}^{n+1} \to \mathbb{R}$, $\pi_2 : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are projections given by

$$\pi_1(t,y) = t \quad \text{and} \quad \pi_2(t,y) = y.$$

For all $s \in [0,1]$ we denote $\psi_s(.) = \psi(.,s)$, so $\psi_0(.) \equiv \mathrm{Id}_\Gamma$, i.e. the identity map over Γ, and

$$\Gamma^s = \psi_s(\Gamma), \quad \Gamma^s_T = \psi_s(\Gamma_T).$$

Definition 1.1. We say that a function $\eta \in C^1(\mathbb{R})$ is an entropy for (1.1), with associated entropy flux $q = (q_1, \dots, q_n) \in C^1(\mathbb{R};\mathbb{R}^n)$, when for all $u \in \mathbb{R}$

$$q_j{}'(u) = \eta'(u) f_j'(u) \quad (j = 1, \dots, n). \tag{1.5}$$

We call $F(u) := (\eta(u), q(u))$ an entropy pair. If η is convex, we say that $F(u)$ is a convex entropy pair. Moreover, we say that $F(u)$ is a generalized entropy pair if it is the uniform limit of a sequence of entropy pairs over compact sets.

Since we are concerned with scalar conservation laws, any $\eta \in C^1(\mathbb{R})$ is an entropy. Indeed, it is enough to take

$$q_j(u) := \int_0^u \eta'(\xi) f_j'(\xi)\, d\xi \quad (j = 1, \dots, n),$$

for (1.5) to be satisfied and, one has to mention, we are always considering convex pairs. An important example of a generalized convex entropy pairs are the Kružkov's entropies, i.e., the parameterized family

$$F(u,v) = (|u-v|, \mathrm{sgn}(u-v)[f(u) - f(v)]) \quad \text{for each } v \in \mathbb{R}. \tag{1.6}$$

Definition 1.2. We call $\mathscr{F}(u,v) := (\alpha(u,v), \beta(u,v))$ a boundary entropy pair, if for each $v \in \mathbb{R}$ fixed, $\mathscr{F}(u,v)$ is a convex entropy pair and

$$\alpha(v,v) = \beta(v,v) = \partial_u \alpha(v,v) = 0. \tag{1.7}$$

Analogously, if $\mathscr{F}(u,v)$ is the uniform limit of a sequence of boundary entropy pairs over compact sets, then we call it a generalized boundary entropy pair.

A common example of boundary entropy pairs are given by the quadratic ones. Here, we give an example of a parameterized family of boundary entropy pair, that it will be used in Section 2

$$\mathscr{F}(u,w,v) = \begin{cases} (w-u, f(w) - f(u)) & \text{if } u \leqslant w \leqslant v, \\ (0,0) & \text{if } w \leqslant u \leqslant v, \\ (u-v, f(u) - f(v)) & \text{if } w \leqslant v \leqslant u, \\ (v-u, f(v) - f(u)) & \text{if } u \leqslant v \leqslant w, \\ (0,0) & \text{if } v \leqslant u \leqslant w, \\ (u-w, f(u) - f(w)) & \text{if } v \leqslant w \leqslant u. \end{cases} \tag{1.8}$$

The following definition tells us in which sense a function $u \in L^\infty(Q_T)$ is a weak entropy solution of (1.1)–(1.3).

Definition 1.3. We say that $u \in L^\infty(Q_T)$ is a weak entropy solution of (1.1)–(1.3) if it satisfies

- Conservation law (1.1): For all $\phi \in C_0^\infty(Q_T)$, $\phi \geqslant 0$, and any entropy pair $F(u) = (\eta(u), q(u))$,

$$\int\int_{Q_T} F(u(t,x)) \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \geqslant 0. \tag{1.9}$$

- Initial condition (1.2): For any open $\tilde{\Omega}$ such that $\overline{\tilde{\Omega}} \subset \Omega$,

$$\text{ess} \lim_{t \to 0^+} \int_{\tilde{\Omega}} |u(t,x) - u_0(x)|\, dx = 0. \tag{1.10}$$

- Boundary condition (1.3): For any $\gamma \in L^1(\Gamma_T; \mathscr{H}^n)$, $\gamma \geqslant 0$ $\mathscr{H}^n$-a.e., and any boundary entropy pair $\mathscr{F}(u,v) = (\alpha(u,v), \beta(u,v))$,

$$\text{ess} \lim_{s \to 0^+} \int_{\Gamma_T} \mathscr{F}(u(\psi_s(r)), u_b(r)) \cdot \mathrm{n}_s(\psi_s(r))\, \gamma(r)\, d\mathscr{H}^n(r) \geqslant 0, \tag{1.11}$$

  where n_s is the outward unit normal field defined $\mathscr{H}^n$-a.e. in Γ_T^s.

For convenience we extend the weak entropy solution $u \in L^\infty(Q_T)$ to $u \in L^\infty(Q)$, that is, we set

$$u(t,x) \equiv 0, \quad \text{for all } (t,x) \in Q - Q_T.$$

Remark 1.1. If $u \in L^\infty(Q_T)$ and satisfies (1.9), then the fields

$$E(u) = (u, f(u)) \quad \text{and} \quad F(u) = (\eta(u), q(u))$$

belong to $\mathscr{DM}(D)$ for any bounded open set $D \subset Q$, as a consequence of the Schwartz lemma on nonnegative distributions, see Schwartz [16]. In particular, the normal traces

$$E \cdot \mathrm{n}|_S \quad \text{and} \quad F \cdot \mathrm{n}|_S$$

are defined for any open subset S of the Lipschitz boundary of any open set D, such that $\overline{D} \subset Q$. Moreover, for these divergence-measure fields, i.e., E and F, we have an extension of the Gauss–Green Theorem, see Chen and Frid [2,3].

Next we recall a result in Chen and Frid [2] which provides, as in [14], a more convenient way to express the concept of weak entropy solution of (1.1)–(1.3) as given by Definition 1.3.

Theorem 1.1 (Equivalence theorem). *A function $u \in L^\infty(Q_T)$ satisfies* (1.9)–(1.11) *if and only if, for any boundary entropy pair $\mathscr{F}(u,v) = (\alpha(u,v), \beta(u,v))$, $v \in \mathbb{R}$, exists a constant $M > 0$, such that for any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$, u satisfies*

$$\begin{aligned}
&\int\int_{Q_T} \mathscr{F}(u(t,x), v) \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \\
&+ M \int_{\Gamma_T} |u_b(r) - v| \phi(r)\, d\mathscr{H}^n(r) \\
&+ \int_\Omega \alpha(u_0(x), v)\phi(0,x)\, dx \geqslant 0. \qquad (1.12)
\end{aligned}$$

As in [14], Theorem 1.1 implies a maximum principle for the hyperbolic problem.

Corollary 1.1 (Maximum principle). *Let*

$$(u, u_b, u_0) \in L^\infty(Q_T) \times L^\infty(\Gamma_T; \mathscr{H}^n) \times L^\infty(\Omega),$$

satisfying (1.12). *Then*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{Q_T} u \leqslant \max\left(\operatorname*{ess\,sup}_{\Gamma_T} u_b, \operatorname*{ess\,sup}_{\Omega} u_0 \right), \\
&\operatorname*{ess\,inf}_{Q_T} u \geqslant \min\left(\operatorname*{ess\,inf}_{\Gamma_T} u_b, \operatorname*{ess\,inf}_{\Omega} u_0 \right). \qquad (1.13)
\end{aligned}$$

In particular,

$$\operatorname*{ess\,sup}_{Q_T} |u| \leqslant \max\left(\operatorname*{ess\,sup}_{\Gamma_T} |u_b|, \operatorname*{ess\,sup}_{\Omega} |u_0| \right). \qquad (1.14)$$

Proof. To get rid of the boundary term, we choose in (1.12) $\phi \in C_0^\infty(Q \cap \{t < T\})$, $\phi \geqslant 0$. We get

$$\int\int_{Q_T} \mathscr{F}(u(t,x), v) \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt + \int_\Omega \alpha(u_0(x), v)\phi(0,x)\, dx \geqslant 0.$$

Now, choosing $\phi(t,x) = \zeta(t)$, with $\zeta \in C_0^\infty(-\infty, T)$, $\zeta \geqslant 0$

$$\int\int_{Q_T} \alpha(u(t,x), v)\, dx\, \zeta'(t)\, dt + \zeta(0) \int_\Omega \alpha(u_0(x), v)\, dx \geqslant 0. \qquad (1.15)$$

First we prove $(1.13)_1$. Define, for each $\ell \in \mathbb{N}$, the following boundary entropy

$$\alpha_\ell(u,v) := [(\max(u - v, 0))^2 + (1/\ell)^2]^{1/2} - 1/\ell,$$

where

$$v := \max\left\{\operatorname*{ess\,sup}_{\Gamma_T} u_b, \operatorname*{ess\,sup}_{\Omega} u_0\right\}.$$

With this boundary entropy in (1.15) and observing that $\alpha_\ell \equiv 0$ in Ω, letting $\ell \to \infty$, we arrive at

$$\int_0^T \int_{\Omega_t} \max(u(t,x) - v, 0)\, dx\, \zeta'(t)\, dt \geqslant 0.$$

Now, let $\delta > 0$, $\tau < T$ and $\zeta(t) = \chi_{(-\tau-\delta,\tau+\delta)}(t)$. Then, after mollifying and passing to the limit, and making $\delta \to 0^+$, we obtain

$$\int_{\Omega_\tau} \max(u(\tau,x) - v, 0)\, dx \leqslant 0.$$

Denoting $z(t,x) = u(t,x) - v$, we get for almost all $t \in (0,T)$

$$||z^+(t)||_{L^1(\Omega_t)} = 0,$$

hence $z^+ = 0$ almost everywhere in Q_T. Consequently,

$$\operatorname*{ess\,sup}_{Q_T} u \leqslant v.$$

Analogously, we obtain $(1.13)_2$. The proof of (1.14) is immediate from (1.13), observing that

$$-\operatorname*{ess\,inf}_{Q_T} u \leqslant -\min\left(\operatorname*{ess\,inf}_{\Gamma_T} u_b, \operatorname*{ess\,inf}_{\Omega} u_0\right),$$

or

$$\operatorname*{ess\,sup}_{Q_T} (-u) \leqslant \max\left(\operatorname*{ess\,sup}_{\Gamma_T} (-u_b), \operatorname*{ess\,sup}_{\Omega} (-u_0)\right). \qquad \square$$

2. Stability

The main focus of this section is to show the L^1-contraction property, which establishes stability for (1.1)–(1.3). Basically we double variables as Kruzkov [11] to obtain this contraction, but since we are concerned with IBVP and the domain is noncylindrical, some features are needed, we have made all of them in details. We prove the continuous dependence of a solution with the initial-boundary data considering Γ_T smooth. In particular, we obtain uniqueness of solution for equal data.

Theorem 2.1 (Stability theorem). *Let Q_T be an open smooth subset of* $\mathbb{R}^{n+1}$, $f \in C^1(\mathbb{R}; \mathbb{R}^n)$ *and F be defined by* (1.6). *Let*

$$(u_i, u_{b_i}, u_{0_i})_{i=1,2} \in L^\infty(Q_T) \times L^\infty(\Gamma_T; \mathscr{H}^n) \times L^\infty(\Omega)$$

satisfying (1.1)–(1.3). *Then there exists positive constants L, M, such that for any function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$, $\phi \geqslant 0$*

$$\int\int_{Q_T} F(u_1, u_2) \cdot \nabla_{t,x}\phi \, dx\, dt + L \int\int_{Q_T} |u_1 - u_2| \phi(t,x)\, dx\, dt \\ + M \int_{\Gamma_T} |u_{b_1} - u_{b_2}| \phi(r)\, d\mathscr{H}^n(r) + \int_\Omega |u_{0_1} - u_{0_2}| \phi(0)\, dx \geqslant 0. \tag{2.1}$$

Moreover, for almost all $t \in (0, T)$ we have

$$\int_{\Omega_t} |u_1(t) - u_2(t)|\, dx \leqslant [1 + Lte^{Lt}] \Bigg\{ \int_\Omega |u_{0_1} - u_{0_2}|\, dx \\ + M \int_0^t \int_{\partial\Omega_\tau} |u_{b_1} - u_{b_2}|\, d\mathscr{H}^{n-1}(r')\, d\tau \Bigg\}. \tag{2.2}$$

Proof. 1. Let $(u_i, u_{b_i}, u_{0_i})_{i=1,2} \in L^\infty(Q_T) \times L^\infty(\Gamma_T; \mathscr{H}^n) \times L^\infty(\Omega)$, satisfying (1.9)–(1.11) in the sense of Definition 1.3. In order to simplify the notation, we drop the subscript $(i = 1, 2)$ in all remaining proof, as soon as indifferent. Now, for each $v, w \in \mathbb{R}$ and $\ell \in \mathbb{N}$, we define

$$F_\ell(u, v) = (\eta_\ell(u, v), q_\ell(u, v)),$$

$$\mathscr{F}_\ell(u, w, v) = (\alpha_\ell(u, w, v), \beta_\ell(u, w, v)),$$

where

$$\eta_\ell(u, v) := [(u - v)^2 + (1/\ell)^2]^{1/2} - 1/\ell,$$

$$q_\ell(u, v) := \int_v^u {\eta_\ell}'(\xi) f'(\xi)\, d\xi,$$

and with $\mathscr{I}[a, b] = [\min\{a, b\}, \max\{a, b\}]$

$$\alpha_\ell(u, w, v) := [(\operatorname{dist}(u, \mathscr{I}[w, v]))^2 + (1/\ell)^2]^{1/2} - 1/\ell,$$

$$\beta_\ell(u, w, v) := \int_w^u \partial_\xi \alpha_\ell(\xi, w, v) f'(\xi)\, d\xi.$$

Hence F_ℓ and $\mathscr{F}_\ell$ converge uniformly to F in (1.6) and $\mathscr{F}$ in (1.8), respectively. Using these entropies and the Dominated Convergence Theorem, from (1.9) and (1.11) we obtain:

For any nonnegative function $\phi \in C_0^\infty(Q_T)$

$$\int\int_{Q_T} F(u(t,x),v)\cdot\nabla_{t,x}\phi(t,x)\,dx\,dt \geqslant 0. \tag{2.3}$$

For any function $\gamma \in L^1(\Gamma_T;\mathscr{H}^n)$, $\gamma \geqslant 0$ $\mathscr{H}^n$-a.e.

$$\operatorname*{ess\,lim}_{s\to 0^+} \int_{\Gamma_T} \mathscr{F}(u(\psi_s(r)),u_b(r),v)\cdot \mathrm{n}_s(\psi_s(r))\,\gamma(r)\,d\mathscr{H}^n(r) \geqslant 0. \tag{2.4}$$

2. Let $h:\mathbb{R}^{n+1}\to[0,1]$ be defined by setting $h(t,x)=s$ if $(t,x)\in\Gamma^s$, $h(t,x)=0$ if $(t,x)\notin Q$ and $h(t,x)=1$ otherwise. So $h(t,x)$ is globally Lipschitz and $\nabla_{t,x}h$ is parallel to n_s $\mathscr{H}^n$-a.e. over Γ^s. In (2.3), we choose

$$\phi(t,x)=\varphi(t,x)(1-\zeta(h(t,x))),$$

where $\varphi\in C_0^\infty((0,T)\times\mathbb{R}^n)$, $\varphi\geqslant 0$ and $\zeta=\chi_{(-\delta,\delta)}$ with $0<\delta<1$. Since $\operatorname{spt}[1-\zeta(h(t,x))]\subset Q$, we have

$$\operatorname{spt}\phi(t,x)\subset\operatorname{spt}\varphi(t,x)\cap\operatorname{spt}[1-\zeta(h(t,x))]\subset Q_T.$$

Then mollifying ζ, i.e., $\zeta_n\equiv\zeta*\rho_n$, where ρ_n is a standard mollifier, we get

$$\begin{aligned}&\int\int_{Q_T} F(u(t,x),v)\cdot\nabla_{t,x}\varphi(t,x)\,dx\,dt\\&-\int\int_{Q_T} F(u(t,x),v)\cdot\nabla_{t,x}h(t,x)\,\varphi(t,x)\,\zeta_n{}'(h(t,x))\,1_{\{|\nabla h|>0\}}\,dx\,dt\geqslant 0.\end{aligned}$$

Using the coarea formula [6,7] in the second integral of the above inequality, we obtain

$$\begin{aligned}&\int\int_{Q_T} F(u(t,x),v)\cdot\nabla_{t,x}\varphi(t,x)\,dx\,dt\\&+\int_0^\delta\int_{\Gamma^s} F(u(r),v)\cdot\mathrm{n}_s(r)\,\varphi(r)\,d\mathscr{H}^n(r)\,\zeta_n{}'(s)\,ds\geqslant 0.\end{aligned}$$

At the same integral we use the area formula [6,7]. Then passing to the limit as $n\to\infty$ and making $\delta\to 0^+$, first we observe that $\varphi|_{\Gamma^s}$ can be replaced by $\varphi|_{\Gamma_T}\circ\psi_s^{-1}$ with an error that goes to 0 when $\delta\to 0^+$, we obtain

$$\begin{aligned}&\int\int_{Q_T} F(u(t,x),v)\cdot\nabla_{t,x}\varphi(t,x)\,dx\,dt\\&-\operatorname*{ess\,lim}_{s\to 0^+}\int_{\Gamma_T} F(u(\psi_s(r)),v)\cdot\mathrm{n}_s(\psi_s(r))\,\varphi(r)\,d\mathscr{H}^n(r)\geqslant 0.\end{aligned} \tag{2.5}$$

Now, we observe that

$$2\mathscr{F}(u,w,v) \equiv F(u,w) + F(u,v) - F(w,v),$$

then from (2.4) we have

$$\begin{aligned}
&\text{ess}\lim_{s\to 0^+} \left\{ \int_{\Gamma_T} F(u(\psi_s(r)), u_b(r)) \cdot \mathrm{n}_s(\psi_s(r))\, \varphi(r)\, d\mathscr{H}^n(r) \right.\\
&+ \int_{\Gamma_T} F(u(\psi_s(r)), v) \cdot \mathrm{n}_s(\psi_s(r))\, \varphi(r)\, d\mathscr{H}^n(r)\\
&\left. - \int_{\Gamma_T} F(u_b(r), v) \cdot \mathrm{n}_s(\psi_s(r))\, \varphi(r)\, d\mathscr{H}^n(r) \right\} \geqslant 0.
\end{aligned}$$

Since the essential limit of each integral exists and denoting by $\mathrm{n}(r)$ the limit of $\mathrm{n}_s(\psi_s(r))$ when $s \to 0^+$, we obtain

$$\begin{aligned}
&- \text{ess}\lim_{s\to 0^+} \int_{\Gamma_T} F(u(\psi_s(r)), v) \cdot \mathrm{n}_s(\psi_s(r))\, \varphi(r)\, d\mathscr{H}^n(r)\\
&\leqslant \text{ess}\lim_{s\to 0^+} \int_{\Gamma_T} F(u(\psi_s(r)), u_b(r)) \cdot \mathrm{n}_s(\psi_s(r))\, \varphi(r)\, d\mathscr{H}^n(r)\\
&\quad - \int_{\Gamma_T} F(u_b(r), v) \cdot \mathrm{n}(r)\, \varphi(r)\, d\mathscr{H}^n(r).
\end{aligned} \tag{2.6}$$

Let $\Theta_{ij} \in L^\infty(\Gamma_T; \mathscr{H}^n)$ such that for any $\gamma \in L^1(\Gamma_T; \mathscr{H}^n)$, $\gamma \geqslant 0$ $\mathscr{H}^n$-a.e.

$$\begin{aligned}
&\text{ess}\lim_{s\to 0^+} \int_{\Gamma_T} F(u_i(\psi_s(r)), u_{b_j}(r)) \cdot \mathrm{n}_s(\psi_s(r))\, \gamma(r)\, d\mathscr{H}^n(r)\\
&\quad = \int_{\Gamma_T} \Theta_{ij}(r)\, \gamma(r)\, d\mathscr{H}^n(r).
\end{aligned} \tag{2.7}$$

Then from (2.5)–(2.7), we obtain for any $\varphi \in C_0^\infty((0,T) \times \mathbb{R}^n)$, $\varphi \geqslant 0$

$$\begin{aligned}
&\int\int_{Q_T} F(u(t,x), v) \cdot \nabla_{t,x}\varphi(t,x)\, dx\, dt\\
&\quad + \int_{\Gamma_T} \Theta(r)\, \varphi(r)\, d\mathscr{H}^n(r) - \int_{\Gamma_T} F(u_b(r), v) \cdot \mathrm{n}(r)\, \varphi(r)\, d\mathscr{H}^n(r) \geqslant 0.
\end{aligned} \tag{2.8}$$

3. In order to make the doubling of variables, we make two changes of coordinates. Let $\mathscr{L} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a diffeomorphism of class C^∞, such that

$$\mathscr{L}(t,y) := \begin{cases} t = t, \\ x = x(t,y), \end{cases}$$

and $Q_T = \mathscr{L}((0,T)\times\Omega)$. For convenience we consider the inclusion map

$$i_{\rm c} : \mathbb{R}\times\partial\Omega \to \mathbb{R}^{n+1},$$

such that for each $(t,y')\mapsto(t,y)$, which is a diffeomorphism over its image. We define $\mathscr{L}_{\rm c} := \mathscr{L}\circ i_{\rm c}$. Let for $(t,x)\in Q_T$

$$u(t,x) = u(\mathscr{L}(t,y)) =: \overline{u}(t,y),$$

$$\varphi(t,x) = \varphi(\mathscr{L}(t,y)) =: \overline{\varphi}(t,y),$$

and for $r = \mathscr{L}_{\rm c}(t,y')\in\Gamma_T$

$$u_b(r) = u_b(\mathscr{L}_{\rm c}(t,y')) =: \overline{u}_b(t,y'),$$

$$\varphi(r) = \varphi(\mathscr{L}_{\rm c}(t,y')) =: \overline{\varphi}(t,y'),$$

analogously $\overline{\Theta}(t,y')$ and $\overline{\mathrm{n}}(t,y')$. Then applying the area formula, we obtain from (2.8)

$$\int_0^T \int_\Omega \left[(D\mathscr{L}(t,y))^{-1}\, F(\overline{u}(t,y),v)\right]\cdot\nabla_{t,y}\overline{\varphi}(t,y) J\mathscr{L}(t,y)\, dy\, dt$$

$$+ \int_0^T \int_{\partial\Omega} \overline{\Theta}(t,y')\overline{\varphi}(t,y')\, J\mathscr{L}_{\rm c}(t,y')\, d\mathscr{H}^{n-1}(y')\, dt$$

$$- \int_0^T \int_{\partial\Omega} F(\overline{u}_b(t,y'),v)\cdot\overline{\mathrm{n}}(t,y')\overline{\varphi}(t,y')\, J\mathscr{L}_{\rm c}(t,y')\, d\mathscr{H}^{n-1}(y')\, dt \geqslant 0,$$

where $J\mathscr{L}$, $J\mathscr{L}_{\rm c}$ are the Jacobians of the transformations. Now, we study the integrand in the first integral, we have

$$[D\mathscr{L}(t,y)]^{-1}F(\overline{u}(t,y),v)$$
$$= \{|\overline{u}-v|, \mathrm{sgn}\,(\overline{u}-v)[\overline{f}(t,y,\overline{u}) - \overline{f}(t,y,v)]\},$$

where for $(t,y)\in(0,T)\times\Omega$ and $i,j = 1,\dots,n$

$$\overline{f}_i(t,y,v) := a_i(t,y)\,v + A_{ij}(t,y)\,f_j(v),$$

$$a_i = \frac{C_{1,i+1}}{C_{11}},\quad A_{ij} = \frac{C_{j+1,i+1}}{C_{11}},\quad C = \mathrm{cof}\,[D\mathscr{L}].$$

Hence we obtain with the first transformation

$$
\begin{aligned}
&\int_0^T \int_\Omega \{|\overline{u}(t,y) - v|\overline{\varphi}_t \\
&+ \operatorname{sgn}(\overline{u}(t,y) - v)[\overline{f}_i(t,y,\overline{u}(t,y)) - \overline{f}_i(t,y,v)]\overline{\varphi}_{y_i}\} J\mathscr{L}(t,y)\, dy\, dt \\
&+ \int_0^T \int_{\partial\Omega} \overline{\Theta}(t,y')\overline{\varphi}(t,y')\, J\mathscr{L}_{\mathrm{c}}(t,y')\, d\mathscr{H}^{n-1}(y')\, dt \\
&- \int_0^T \int_{\partial\Omega} F(\overline{u}_b(t,y'),v)\cdot \overline{\mathrm{n}}(t,y')\overline{\varphi}(t,y')\, J\mathscr{L}_{\mathrm{c}}(t,y')\, d\mathscr{H}^{n-1}(y')\, dt \geqslant 0. \qquad (2.9)
\end{aligned}
$$

Now we make a second transformation. Since $\partial\Omega$ is smooth, for any $y_0 \in \partial\Omega$ we can find $r > 0$ and a mapping $\gamma \in C^\infty(\mathbb{R}^{n-1};\mathbb{R})$ such that, upon rotating and relabelling the coordinate axes if necessary

$$\Omega \cap U(y_0,r) = \{y \mid y_n > \gamma(y_1,\dots,y_{n-1})\} \cap U(y_0,r).$$

Write $U \equiv U(y_0,r)$ and suppose temporally that $\overline{\varphi}(t,y)$ has compact support contained in $(0,T)\times U$. Let

$$\begin{cases} y_i = z_i =: H_i(z) \quad (i = 1,\dots,n-1), \\ y_n = z_n - \gamma(z_1,\dots,z_{n-1}) =: H_n(z), \end{cases}$$

and denote $y = (y', y_n) = H(z)$. Analogously, we have

$$\begin{cases} z_i = y_i =: (H^{-1})_i(y) \quad (i = 1,\dots,n-1), \\ z_n = y_n + \gamma(y_1,\dots,y_{n-1}) =: (H^{-1})_n(y), \end{cases}$$

and $z = (z', z_n) = H^{-1}(y)$. Moreover, we define

$$B \coloneqq H^{-1}(U), \quad B_+ \coloneqq H^{-1}(U\cap\Omega) \quad \text{and} \quad B_0 \coloneqq H^{-1}(U\cap\partial\Omega),$$

and consider the following change of variables:

$$\mathscr{H}(t,z) \coloneqq \begin{cases} t = t, \\ y = H(z). \end{cases}$$

Then $\mathscr{H}$ is a diffeomorphism of class C^∞, and for all $(t,z)\in(0,T)\times B$ the Jacobian of $\mathscr{H}$ is one. Denote

$$
\begin{aligned}
&J(t,z) = (J\mathscr{L}\circ\mathscr{H})(t,z), \\
&\tilde{\Theta}(t,z') = \overline{\Theta}(\mathscr{H}(t,z'))\,(J\mathscr{L}_{\mathrm{c}}/J\mathscr{L})(\mathscr{H}(t,z')), \\
&\tilde{\mathrm{n}}(t,z') = \overline{\mathrm{n}}(\mathscr{H}(t,z'))\,(J\mathscr{L}_{\mathrm{c}}/J\mathscr{L})(\mathscr{H}(t,z')),
\end{aligned}
$$

where $(t,z') \equiv (t,z',0)$ on $(0,T)\times B_0$. Finally,

$$\overline{u}(t,y) = \overline{u}(\mathscr{H}(t,z)) =: \tilde{u}(t,z),$$

$$\overline{\varphi}(t,y) = \overline{\varphi}(\mathscr{H}(t,z)) =: \tilde{\varphi}(t,z),$$

where $\tilde{\varphi}(t,z)$ has compact support contained in $(0,T)\times B$. Remaking the same procedure for the first transformation, from (2.9) we obtain

$$\begin{aligned}
&\int_0^T \int_{B_+} \{|\tilde{u}(t,z)-v|\tilde{\varphi}_t \\
&+ \operatorname{sgn}(\tilde{u}(t,z)-v)[\tilde{f}_i(t,z,\tilde{u}(t,z)) - \tilde{f}_i(t,z,v)]\tilde{\varphi}_{z_i}\}J(t,z)\,dz\,dt \\
&+ \int_0^T \int_{B_0} \tilde{\Theta}(t,z')\tilde{\varphi}(t,z')\,J(t,z')\,dz'\,dt \\
&- \int_0^T \int_{B_0} F(\tilde{u}_b(t,z'),v)\cdot \tilde{\mathrm{n}}(t,z')\tilde{\varphi}(t,z')\,J(t,z')\,dz'\,dt \geqslant 0.
\end{aligned} \tag{2.10}$$

4. Further we make $\phi(t,z) = \tilde{\varphi}(t,z)J(t,z)$, so $\phi(t,z)$ is a nonnegative smooth function and has compact support contained in $(0,T)\times B$. Then from (2.10) we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n_+} \{|\tilde{u}(t,z)-v|\phi_t \\
&\quad + \operatorname{sgn}(\tilde{u}(t,z)-v)[\tilde{f}_i(t,z,\tilde{u}(t,z)) - \tilde{f}_i(t,z,v)]\phi_{z_i}\}\,dz\,dt \\
&\quad - \int_0^T \int_{\mathbb{R}^n_+} \operatorname{sgn}(\tilde{u}(t,z)-v)\tilde{f}_{i_{z_i}}(t,z,v)\phi\,dz\,dt \\
&\quad - \int_0^T \int_{\mathbb{R}^n_+} \operatorname{sgn}(\tilde{u}(t,z)-v)g(t,z,\tilde{u}(t,z),v)\phi\,dz\,dt \\
&\quad + \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}(t,z')\phi\,dz'\,dt \\
&\quad - \int_0^T \int_{\mathbb{R}^{n-1}} F(\tilde{u}_b(t,z'),v)\cdot \tilde{\mathrm{n}}(t,z')\phi\,dz'\,dt \geqslant 0,
\end{aligned} \tag{2.11}$$

where the function g is given by

$$\begin{aligned}
&g(t,z,\tilde{v},v) \\
&\quad := J^{-1}(t,z)[(\tilde{v}-v),(\tilde{f}(t,z,\tilde{v}) - \tilde{f}(t,z,v))]\cdot \nabla_{t,z}J - \tilde{f}_{i_{z_i}}(t,z,v).
\end{aligned}$$

5. Now, we double the variables. Let $\rho \in C_0^\infty(\mathbb{R}^{n+1})$ be a symmetric mollifier, then given $\varepsilon > 0$ there exists a constant $C > 0$ such that, for all $(t,z) \in \mathbb{R}^{n+1}$

$$|\rho_\varepsilon(t,z)| \leqslant \frac{C}{\varepsilon^{n+1}}, \ |\nabla_{t,z}\rho_\varepsilon(t,z)| \leqslant \frac{C}{\varepsilon^{n+2}}.$$

For each $(t,z), (\tau,\zeta) \in (0,T) \times \mathbb{R}^n_+$ and $\phi \in C_0^\infty((0,T) \times \mathbb{R}^n)$, $\phi \geqslant 0$, spt $\phi \subset (0,T) \times B$ we set

$$\phi_\varepsilon(t,z,\tau,\zeta) := \phi\left(\frac{t+\tau}{2}, \frac{z+\zeta}{2}\right)\rho_\varepsilon(t-\tau, z-\zeta).$$

Hold $(\tau,\zeta) \in (0,T) \times \mathbb{R}^n_+$ fixed and replace in (2.11) $\tilde{u}, \tilde{u}_b$ by $\tilde{u}_1, \tilde{u}_{b_1}$, v by $\tilde{u}_2(\tau,\zeta)$ and ϕ by ϕ_ε. After integration over $(\tau,\zeta) \in (0,T) \times \mathbb{R}^n_+$, we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \frac{1}{2}\Big\{|\tilde{u}_1(t,z) - \tilde{u}_2(\tau,\zeta)|\phi_t\left(\frac{t+\tau}{2}, \frac{z+\zeta}{2}\right) \\
&\quad + \operatorname{sgn}(\tilde{u}_1(t,z) - \tilde{u}_2(\tau,\zeta))[\tilde{f}_i(t,z,\tilde{u}_1(t,z)) \\
&\quad - \tilde{f}_i(t,z,\tilde{u}_2(\tau,\zeta))]\phi_{z_i}\left(\frac{t+\tau}{2}, \frac{z+\zeta}{2}\right)\Big\}\rho_\varepsilon(t-\tau, z-\zeta)\, dz\, dt\, d\zeta\, d\tau \\
&+ \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \{|\tilde{u}_1(t,z) - \tilde{u}_2(\tau,\zeta)|\rho_{\varepsilon_t}(t-\tau, z-\zeta) \\
&\quad + \operatorname{sgn}(\tilde{u}_1(t,z) - \tilde{u}_2(\tau,\zeta))[\tilde{f}_i(t,z,\tilde{u}_1(t,z)) \\
&\quad - \tilde{f}_i(t,z,\tilde{u}_2(\tau,\zeta))]\rho_{\varepsilon_{z_i}}(t-\tau, z-\zeta)\}\phi\left(\frac{t+\tau}{2}, \frac{z+\zeta}{2}\right) dz\, dt\, d\zeta\, d\tau \\
&- \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \operatorname{sgn}(\tilde{u}_1(t,z) - \tilde{u}_2(\tau,\zeta))\tilde{f}_{i_{z_i}}(t,z,\tilde{u}_2(\tau,\zeta)) \\
&\quad \times \phi\left(\frac{t+\tau}{2}, \frac{z+\zeta}{2}\right)\rho_\varepsilon(t-\tau, z-\zeta)\, dz\, dt\, d\zeta\, d\tau \\
&- \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \operatorname{sgn}(\tilde{u}_1(t,z) - \tilde{u}_2(\tau,\zeta))g(t,z,\tilde{u}_1(t,z), \tilde{u}_2(\tau,\zeta)) \\
&\quad \times \phi\left(\frac{t+\tau}{2}, \frac{z+\zeta}{2}\right)\rho_\varepsilon(t-\tau, z-\zeta)\, dz\, dt\, d\zeta\, d\tau
\end{aligned}$$

$$+\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{11}(t,z')\phi\left(\frac{t+\tau}{2},\frac{z'+\zeta}{2}\right)\rho_\varepsilon(t-\tau,z'-\zeta)\,dz'\,dt\,d\zeta\,d\tau$$

$$-\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} F(\tilde{u}_{b_1}(t,z'),\tilde{u}_2(\tau,\zeta))\cdot\tilde{\mathrm{n}}(t,z')$$

$$\times\,\phi\left(\frac{t+\tau}{2},\frac{z'+\zeta}{2}\right)\rho_\varepsilon(t-\tau,z-\zeta)\,dz'\,dt\,d\zeta\,d\tau\geqslant 0.$$

Observing that $\nabla_{\tau,\zeta}\phi=\nabla_{t,z}\phi$, $\nabla_{\tau,\zeta}\rho_\varepsilon=-\nabla_{t,z}\rho_\varepsilon$ and adding the analogous equation where $\tilde{u}_1$ and $\tilde{u}_2$ have been interchanged and so (t,z) with (τ,ζ), we obtain

$$\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \rho_\varepsilon\{|\tilde{u}_1(t,z)-\tilde{u}_2(\tau,\zeta)|\phi_t+\frac{1}{2}\operatorname{sgn}(\tilde{u}_1(t,z)-\tilde{u}_2(\tau,\zeta))$$

$$\times\,[(\tilde{f}_i(t,z,\tilde{u}_1)+\tilde{f}_i(\tau,\zeta,\tilde{u}_1))-(\tilde{f}_i(t,z,\tilde{u}_2)+\tilde{f}_i(\tau,\zeta,\tilde{u}_2))]\phi_{z_i}\}\,dz\,dt\,d\zeta\,d\tau$$

$$+\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \phi\operatorname{sgn}(\tilde{u}_1(t,z)-\tilde{u}_2(\tau,\zeta))$$

$$\times\,\{[\tilde{f}_i(t,z,\tilde{u}_1)-\tilde{f}_i(\tau,\zeta,\tilde{u}_1)]\rho_{\varepsilon_{z_i}}+\rho_\varepsilon\tilde{f}_{i_{\zeta_i}}(\tau,\zeta,\tilde{u}_1)\}\,dz\,dt\,d\zeta\,d\tau$$

$$+\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \phi\operatorname{sgn}(\tilde{u}_1(t,z)-\tilde{u}_2(\tau,\zeta))$$

$$\times\,\{[\tilde{f}_i(\tau,\zeta,\tilde{u}_2)-\tilde{f}_i(t,z,\tilde{u}_2)]\rho_{\varepsilon_{\zeta_i}}+\rho_\varepsilon\tilde{f}_{i_{z_i}}(t,z,\tilde{u}_2)\}\,dz\,dt\,d\zeta\,d\tau$$

$$-\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \operatorname{sgn}(\tilde{u}_1(t,z)-\tilde{u}_2(\tau,\zeta))$$

$$\times\,[g(t,z,\tilde{u}_1,\tilde{u}_2)-g(\tau,\zeta,\tilde{u}_2,\tilde{u}_1)]\phi\rho_\varepsilon\,dz\,dt\,d\zeta\,d\tau$$

$$+\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{11}(t,z')\phi\left(\frac{t+\tau}{2},\frac{z'+\zeta}{2}\right)\rho_\varepsilon(t-\tau,z'-\zeta)\,dz'\,dt\,d\zeta\,d\tau$$

$$-\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} F(\tilde{u}_{b_1}(t,z'),\tilde{u}_2(\tau,\zeta))\cdot\tilde{\mathrm{n}}(t,z')$$

$$\times\,\phi\left(\frac{t+\tau}{2},\frac{z'+\zeta}{2}\right)\rho_\varepsilon(t-\tau,z'-\zeta)\,dz'\,dt\,d\zeta\,d\tau$$

$$+\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{22}(\tau,\zeta')\phi\left(\frac{t+\tau}{2},\frac{z+\zeta'}{2}\right)\rho_\varepsilon(t-\tau,z-\zeta')\,d\zeta'\,d\tau\,dz\,dt$$
$$-\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} F(\tilde{u}_1(t,z),\tilde{u}_{b_2}(\tau,\zeta'))\cdot\tilde{\mathrm{n}}(\tau,\zeta')$$
$$\times\phi\left(\frac{t+\tau}{2},\frac{z+\zeta'}{2}\right)\rho_\varepsilon(t-\tau,z-\zeta')\,d\zeta'\,d\tau\,dz\,dt$$
$$\equiv I_1+I_2+I_3-I_4+I_5-I_6+I_7-I_8\geqslant 0. \tag{2.12}$$

Now, we are about to study the convergence of each term. First we show that

$$\lim_{\varepsilon\to 0^+} I_1 = \int_0^T \int_{\mathbb{R}^n_+} |\tilde{u}_1(t,z)-\tilde{u}_2(t,z)|\phi_t(t,z) + \operatorname{sgn}(\tilde{u}_1-\tilde{u}_2)[\tilde{f}_i(t,z,\tilde{u}_1(t,z))-\tilde{f}_i(t,z,\tilde{u}_2(t,z))]\phi_{z_i}(t,z)\,dz\,dt. \tag{2.13}$$

Let

$$\mathbb{F}(t,z,\tau,\zeta,\tilde{v}_1,\tilde{v}_2) := |\tilde{v}_1-\tilde{v}_2|\phi_t\left(\frac{t+\tau}{2},\frac{z+\zeta}{2}\right) + \frac{1}{2}\operatorname{sgn}(\tilde{v}_1-\tilde{v}_2)\{[\tilde{f}_i(t,z,\tilde{v}_1)+\tilde{f}_i(\tau,\zeta,\tilde{v}_1)] - [\tilde{f}_i(t,z,\tilde{v}_2)+\tilde{f}_i(\tau,\zeta,\tilde{v}_2)]\}\phi_{z_i}\left(\frac{t+\tau}{2},\frac{z+\zeta}{2}\right).$$

Then $\mathbb{F}$ is Lipschitz in all variables, see Kružkov [11, Lemma 3], and as usual we make

$$I_1=\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \{\mathbb{F}(t,z,\tau,\zeta,\tilde{u}_1(t,z),\tilde{u}_2(\tau,\zeta)) - \mathbb{F}(t,z,t,z,\tilde{u}_1(t,z),\tilde{u}_2(\tau,\zeta)) + \mathbb{F}(t,z,t,z,\tilde{u}_1(t,z),\tilde{u}_2(\tau,\zeta)) - \mathbb{F}(t,z,t,z,\tilde{u}_1(t,z),\tilde{u}_2(t,z))\}\rho_\varepsilon(t-\tau,z-\zeta)\,dz\,dt\,d\zeta\,d\tau$$
$$+\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \mathbb{F}(t,z,\tilde{u}_1(t,z),\tilde{u}_2(t,z))\rho_\varepsilon(t-\tau,z-\zeta)\,dz\,dt\,d\zeta\,d\tau$$
$$\equiv J_1+J_2.$$

From the properties of $\mathbb{F}$ and ρ_ε, we get

$$|J_1|\leqslant C[\varepsilon+\frac{1}{\varepsilon^{n+1}}\int\int\int\int_{\|(t,z)-(\tau,\zeta)\|\leqslant\varepsilon} |\tilde{u}_2(\tau,\zeta)-\tilde{u}_2(t,z)|\,dz\,dt\,d\zeta\,d\tau],$$

where C is a positive constant independent of ε. Then $J_1 \to 0$ as $\varepsilon \to 0^+$. Moreover, for $\varepsilon > 0$ sufficiently small J_2 is independent of ε. Indeed, making the change of coordinates $(t, z, \tau, \zeta) \mapsto (t, z, \varsigma = t - \tau, \xi = z - \zeta)$, it follows that

$$
\begin{aligned}
J_2 &= \int_0^T \int_{\mathbb{R}^n_+} \mathbb{F}(t, z, \tilde{u}_1(t, z), \tilde{u}_2(t, z)) \left[\int \int_{\mathbb{R}^{n+1}} \rho_\varepsilon(\varsigma, \xi) \, d\xi \, d\varsigma \right] dz \, dt \\
&= \int_0^T \int_{\mathbb{R}^n_+} \mathbb{F}(t, z, \tilde{u}_1(t, z), \tilde{u}_2(t, z)) \, dz \, dt.
\end{aligned}
$$

From what (2.13) follows. Now, we show that I_2 and I_3 go to zero when $\varepsilon \to 0^+$. Since $\tilde{f}$ is a smooth function in the first two variables, we can write

$$
\begin{aligned}
&[\tilde{f}_i(t, z, \tilde{v}) - \tilde{f}_i(\tau, \zeta, \tilde{v})] \rho_{\varepsilon_{z_i}} + \tilde{f}_{i_{\zeta_i}}(\tau, \zeta, \tilde{v}) \rho_\varepsilon \\
&\quad = \tilde{f}_{i_\tau}(\tau, \zeta, \tilde{v}) \, (t - \tau) \, \rho_{\varepsilon_{z_i}} \\
&\qquad + \tilde{f}_{i_{\zeta_j}}(\tau, \zeta, \tilde{v}) \, (z_j - \zeta_j) \, \rho_{\varepsilon_{z_i}} + o(\varepsilon^2) \rho_{\varepsilon_{z_i}} + \tilde{f}_{i_{\zeta_i}}(\tau, \zeta, \tilde{v}) \rho_\varepsilon \\
&\quad = \tilde{f}_{i_\tau}(\tau, \zeta, \tilde{v}) [(t - \tau) \rho_\varepsilon]_{z_i} \\
&\qquad + \tilde{f}_{i_{\zeta_j}}(\tau, \zeta, \tilde{v}) [(z_j - \zeta_j) \rho_\varepsilon]_{z_i} + o(\varepsilon^2) \rho_{\varepsilon_{z_i}}.
\end{aligned}
$$

Moreover,

$$
\phi\left(\frac{t + \tau}{2}, \frac{z + \zeta}{2} \right) = \phi(\tau, \zeta) + \nabla_{\tau, \zeta} \phi(\tau, \zeta) \cdot \left(\frac{t - \tau}{2}, \frac{z - \zeta}{2} \right) + o(\varepsilon^2).
$$

Hence,

$$
\begin{aligned}
I_2 = &\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \phi(\tau, \zeta) \operatorname{sgn}(\tilde{u}_1(t, z) - \tilde{u}_2(\tau, \zeta)) \{ f_{i_\tau}(\tau, \zeta, \tilde{u}_1(t, z)) \\
&\times [(t - \tau) \rho_\varepsilon]_{z_i} + \tilde{f}_{i_{\zeta_j}}(\tau, \zeta, \tilde{u}_1(t, z)) [(z_j - \zeta_j) \rho_\varepsilon]_{z_i} \} \, dz \, dt \, d\zeta \, d\tau \\
&+ \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \nabla_{\tau, \zeta} \phi(\tau, \zeta) \cdot \left(\frac{t - \tau}{2}, \frac{z - \zeta}{2} \right) \operatorname{sgn}(\tilde{u}_1(t, z) - \tilde{u}_2(\tau, \zeta)) \\
&\times [\tilde{f}_i(t, z, \tilde{u}_1(t, z)) - \tilde{f}_i(\tau, \zeta, \tilde{u}_1(t, z))] \rho_{\varepsilon_{z_i}} \, dz \, dt \, d\zeta \, d\tau \\
&+ \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \nabla_{\tau, \zeta} \phi(\tau, \zeta) \cdot \left(\frac{t - \tau}{2}, \frac{z - \zeta}{2} \right) \operatorname{sgn}(\tilde{u}_1(t, z) - \tilde{u}_2(\tau, \zeta)) \\
&\times \tilde{f}_{i_{\zeta_i}}(\tau, \zeta, \tilde{u}_1(t, z)) \rho_\varepsilon \, dz \, dt \, d\zeta \, d\tau \\
&+ \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \mathbb{H}(t, z, \tau, \zeta, \tilde{u}_1(t, z), \tilde{u}_2(\tau, \zeta)) \rho_{\varepsilon_{z_i}} \, dz \, dt \, d\zeta \, d\tau \\
\equiv &\; J_3 + J_4 + J_5 + J_6,
\end{aligned}
$$

where the modulus of $\mathbb{H}$ is $o(\varepsilon^2)$ and from the properties of the functions $\tilde{f}$, ρ_ε, we obtain

$$|J_4| \leqslant \frac{C}{\varepsilon^{n+2}}\,\varepsilon\,\varepsilon\,\varepsilon^{n+1} \underset{\varepsilon\to 0^+}{\longrightarrow} 0,$$

$$|J_5| \leqslant \frac{C}{\varepsilon^{n+1}}\,\varepsilon\,\varepsilon^{n+1} \underset{\varepsilon\to 0^+}{\longrightarrow} 0,$$

$$|J_6| \leqslant \frac{o(\varepsilon^2)}{\varepsilon^{n+2}}\,\varepsilon^{n+1} \underset{\varepsilon\to 0^+}{\longrightarrow} 0.$$

Now, for convenience we denote

$$\mathbb{K}_i(\tau,\zeta,\tilde{v}_1,\tilde{v}_2) = \phi(\tau,\zeta)\,\mathrm{sgn}\,(\tilde{v}_1-\tilde{v}_2)\,f_{i_\tau}(\tau,\zeta,\tilde{v}_1),$$

$$\mathbb{K}_{ij}(\tau,\zeta,\tilde{v}_1,\tilde{v}_2) = \phi(\tau,\zeta)\,\mathrm{sgn}\,(\tilde{v}_1-\tilde{v}_2)\,\tilde{f}_{i_{\zeta_j}}(\tau,\zeta,\tilde{v}_1).$$

Like $\mathbb{F}$ the functions $\mathbb{K}_i, \mathbb{K}_{ij}$ are Lipschitz in all variables, and with this notation

$$J_3 = \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \mathbb{K}_i(\tau,\zeta,\tilde{u}_1(t,z),\tilde{u}_2(\tau,\zeta))[(t-\tau)\rho_\varepsilon]_{z_i}$$
$$+\,\mathbb{K}_{ij}(\tau,\zeta,\tilde{u}_1(t,z),\tilde{u}_2(\tau,\zeta))[(z_j-\zeta_j)\rho_\varepsilon]_{z_i}\,dz\,dt\,d\zeta\,d\tau.$$

Since ρ has compact support, integrating by parts we have

$$\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \mathbb{K}_i(\tau,\zeta,\tilde{u}_1(\tau,\zeta),\tilde{u}_2(\tau,\zeta))[(t-\tau)\rho_\varepsilon]_{z_i}$$
$$+\ \mathbb{K}_{ij}(\tau,\zeta,\tilde{u}_1(\tau,\zeta),\tilde{u}_2(\tau,\zeta))[(z_j-\zeta_j)\rho_\varepsilon]_{z_i}\,dz\,dt\,d\zeta\,d\tau \equiv 0.$$

Subtracting the above expression from J_3 and due the properties of $\mathbb{K}_i, \mathbb{K}_{ij}$ and ρ_ε we get

$$|J_3| \leqslant \frac{C\,\varepsilon}{\varepsilon^{n+2}} \int\int\int\int_{\|(t,z)-(\tau,\zeta)\|\leqslant\varepsilon} |\tilde{u}_1(t,z)-\tilde{u}_1(\tau,\zeta)|\,dz\,dt\,d\zeta\,d\tau,$$

from what it follows that $J_3 \to 0$ when $\varepsilon\to 0^+$. Then $I_2 \to 0$ as $\varepsilon \to 0^+$ and analogously I_3. Now, let us look at I_4. Set

$$\mathbb{L}(t,z,\tau,\zeta,\tilde{v}_1,\tilde{v}_2) := \mathrm{sgn}\,(\tilde{v}_1-\tilde{v}_2)$$
$$[g(t,z,\tilde{v}_1,\tilde{v}_2) - g(\tau,\zeta,\tilde{v}_2,\tilde{v}_1)]\,\phi\left(\frac{t+\tau}{2},\frac{z+\zeta}{2}\right),$$

which is a Lipschitz function in all variables. Proceeding analogously for $\mathbb{F}$ in I_1, we get

$$\lim_{\varepsilon\to 0^+} \int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^n_+} \mathbb{L}(t,z,\tau,\zeta,\tilde{u}_1(t,z),\tilde{u}_2(\tau,\zeta))\rho_\varepsilon \, dz\, dt\, d\zeta\, d\tau$$
$$= \int_0^T \int_{\mathbb{R}^n_+} 2\, J^{-1}(t,z)\,\mathrm{sgn}\,(\tilde{u}_1(t,z)-\tilde{u}_2(t,z))[\tilde{u}_1(t,z)-\tilde{u}_2(t,z),$$
$$\tilde{f}(t,z,\tilde{u}_1(t,z))-\tilde{f}(t,z,\tilde{u}_2(t,z))]\cdot\nabla_{t,z}J(t,z)\,\phi(t,z)\,dz\,dt$$
$$+ \int_0^T \int_{\mathbb{R}^n_+} [\tilde{f}_{i_{z_i}}(t,z,\tilde{u}_1(t,z))-\tilde{f}_{i_{z_i}}(t,z,\tilde{u}_2(t,z))]\,\phi(t,z)\,dz\,dt. \tag{2.14}$$

Further we observe that there exists a constant $L>0$ depending on F, $\mathscr{L}$ and $\mathscr{H}$ such that

$$- \int_0^T \int_{\mathbb{R}^n_+} \{J^{-1}(t,z)\mathrm{sgn}\,(\tilde{u}_1(t,z)-\tilde{u}_2(t,z))$$
$$\times[\tilde{u}_1-\tilde{u}_2,\tilde{f}(t,z,\tilde{u}_1)-\tilde{f}(t,z,\tilde{u}_2)]\cdot\nabla_{t,z}J(t,z)$$
$$+ [\tilde{f}_{i_{z_i}}(t,z,\tilde{u}_1)-\tilde{f}_{i_{z_i}}(t,z,\tilde{u}_2)]\}\phi(t,z)\,dz\,dt$$
$$\leqslant L\int_0^T \int_{\mathbb{R}^n_+} |\tilde{u}_1(t,z)-\tilde{u}_2(t,z)|\phi(t,z)\,dz\,dt. \tag{2.15}$$

Now we investigate the boundary terms. We begin showing that

$$\lim_{\varepsilon\to 0^+} I_5 = \frac{1}{2}\int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{11}(t,z')\phi(t,z')\,dz'\,dt. \tag{2.16}$$

From Fubini's Theorem, we can write

$$\left|\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{11}(t,z')\phi\left(\frac{t+\tau}{2},\frac{z'+\zeta}{2}\right)\rho_\varepsilon(t-\tau,z'-\zeta)\,dz'\,dt\,d\zeta\,d\tau\right.$$
$$\left. -1/2\int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{11}(t,z')\phi(t,z')\,dz'\,dt\right|$$
$$\leqslant \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{11}(t,z')$$
$$\left[\int_0^T \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_+} \left|\phi\left(\frac{t+\tau}{2},\frac{z'+\zeta'}{2},\frac{0+\zeta_n}{2}\right)-\phi(t,z',0)\right|\right.$$
$$\left.\rho_\varepsilon(t-\tau,z'-\zeta',0-\zeta_n)\,d\zeta_n\,d\zeta'\,d\tau\right]dz'\,dt$$

$$\leqslant \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{11}(t,z') \frac{C}{\varepsilon^{n+1}}$$
$$\int\int\int_{B_\varepsilon(t,z',0)} |\phi\left(t-\frac{\varsigma}{2}, z'-\frac{\xi'}{2}, 0-\frac{\xi_n}{2}\right) - \phi(t,z',0)|\, d\xi_n\, d\xi'\, d\varsigma\, dz'\, dt.$$

Then taking the limit as $\varepsilon \to 0^+$ we obtain (2.16). Analogously

$$\lim_{\varepsilon\to 0^+} I_7 = \frac{1}{2}\int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{22}(t,z')\phi(t,z')\, dz'\, dt. \tag{2.17}$$

Remain to show the convergence of I_6 and I_8. Let us show that

$$\lim_{\varepsilon\to 0^+} I_6 = \frac{1}{2}\int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{21}(t,z')\phi(t,z')\, dz'\, dt. \tag{2.18}$$

Again by Fubini's Theorem, we have

$$\left|\int_0^T \int_{\mathbb{R}^n_+} \int_0^T \int_{\mathbb{R}^{n-1}} F(\tilde{u}_{b_1}(t,z'), \tilde{u}_2(\tau,\zeta)) \cdot \tilde{\mathrm{n}}(t,z')\, \phi\left(\frac{t+\tau}{2}, \frac{z'+\zeta}{2}\right)\right.$$
$$\left.\rho_\varepsilon(t-\tau, z'-\zeta)\, dz'\, dt\, d\zeta\, d\tau - 1/2 \int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{21}(t,z')\phi(t,z')\, dz'\, dt\right|$$
$$\leqslant \int_0^T \int_{\mathbb{R}^{n-1}} \frac{C}{\varepsilon^{n+1}} \int\int\int_{B_\varepsilon(t,z',0)} \left|\phi\left(t-\frac{\varsigma}{2}, z'-\frac{\xi'}{2}, 0-\frac{\xi_n}{2}\right)\right.$$
$$\left. -\phi(t,z',0)\right|\, d\xi_n\, d\xi'\, d\varsigma\;\, dz'\, dt$$
$$+ \frac{C}{\varepsilon^{n+1}} \int\int \int_{B_\varepsilon(t,z',0)} \left| \int_0^T \int_{\mathbb{R}^{n-1}} [F(\tilde{u}_{b_1}(t,z'), \tilde{u}_2(t-\varsigma, z'-\xi', 0-\xi_n))\right.$$
$$\left. \cdot\, \tilde{\mathrm{n}}(t,z') - \tilde{\Theta}_{21}(t,z')]\phi(t,z')\, dz'\, dt\right|\, d\xi_n\, d\xi'\, d\varsigma,$$

from what (2.18) follows. Analogously

$$\lim_{\varepsilon\to 0^+} I_8 = \frac{1}{2}\int_0^T \int_{\mathbb{R}^{n-1}} \tilde{\Theta}_{12}(t,z')\phi(t,z')\, dz'\, dt. \tag{2.19}$$

Finally, considering (2.13)–(2.19) from (2.12) we obtain

$$\int_0^T \int_{\mathbb{R}^n_+} \{|\tilde{u}_1(t,z) - \tilde{u}_2(t,z)|, \operatorname{sgn}(\tilde{u}_1(t,z) - \tilde{u}_2(t,z))$$
$$\times [\tilde{f}(t,z,\tilde{u}_1(t,z)) - \tilde{f}(t,z,\tilde{u}_2(t,z))]\} \cdot \nabla_{t,z}\phi(t,z)\, dz\, dt$$

$$
\begin{aligned}
&- \int_0^T \int_{\mathbb{R}^n_+} J^{-1}(t,z)\{|\tilde{u}_1(t,z)-\tilde{u}_2(t,z)|, \operatorname{sgn}(\tilde{u}_1(t,z)-\tilde{u}_2(t,z)) \\
&\quad \times [\tilde{f}(t,z,\tilde{u}_1(t,z)) - \tilde{f}(t,z,\tilde{u}_2(t,z))]\} \cdot \nabla_{t,z} J(t,z)\, \phi(t,z)\, dz\, dt \\
&+ L \int_0^T \int_{\mathbb{R}^n_+} |\tilde{u}_1(t,z)-\tilde{u}_2(t,z)| \phi(t,z)\, dz\, dt \\
&+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^{n-1}} \sum_{i,j=1}^{2} (-1)^{i+j} \tilde{\Theta}_{ij}(t,z')\phi(t,z')\, dz'\, dt \geqslant 0. \qquad (2.20)
\end{aligned}
$$

6. Now, we write $\phi(t,z) = \tilde{\varphi}(t,z)J(t,z)$ and make the change of variables returning to the cylinder $(0,T)\times\Omega$. Then, we get

$$
\begin{aligned}
&\int_0^T \int_{\Omega\cap U} [|\overline{u}_1(t,y)-\overline{u}_2(t,y)|, \operatorname{sgn}(\overline{u}_1(t,y)-\overline{u}_2(t,y)) \\
&\quad \times (\overline{f}(t,y,\overline{u}_1(t,y)) - \overline{f}(t,y,\overline{u}_2(t,y)))] \cdot \nabla_{t,y}\overline{\varphi}(t,y)\, J\mathscr{L}(t,y)\, dy\, dt \\
&\quad + L \int_0^T \int_{\Omega\cap U} |\overline{u}_1(t,y)-\overline{u}_2(t,y)| \overline{\varphi}(t,y)\, J\mathscr{L}(t,y)\, dy\, dt \\
&\quad + \frac{1}{2} \int_0^T \int_{\partial\Omega\cap U} \sum_{i,j=1}^{2} (-1)^{i+j} \overline{\Theta}_{ij}(t,y')\overline{\varphi}(t,y')\, J\mathscr{L}_{\mathrm{c}}(t,y')\, d\mathscr{H}^{n-1}(y')\, dt \geqslant 0. \qquad (2.21)
\end{aligned}
$$

Before we make the transformation to the noncylindrical domain, we remember that we have supposed that $\operatorname{spt}\overline{\varphi}\subset(0,T)\times U$. Since $\partial\Omega$ is compact, we can cover $\partial\Omega$ by a finite number of balls $U_i = U(y_{0_i}, r_i), (i=1,\cdots,N)$. Let $\{\zeta_i\}_{i=0}^N$ be a sequence of smooth functions, such that

$$
\begin{cases}
0\leqslant\zeta_i\leqslant 1, \quad \operatorname{spt}\zeta_i\subset U_i \quad (i=1,\ldots,N) \\
0\leqslant\zeta_0\leqslant 1, \quad \operatorname{spt}\zeta_0\subset U_0\subset\Omega \\
\sum_{i=0}^N \zeta_i = 1 \quad \text{on } \Omega\subset\bigcup_{i=0}^N U_i.
\end{cases}
$$

Given $\overline{\varphi}\in C_0^\infty((0,T)\times\mathbb{R}^n)$, we make

$$
\overline{\varphi}_i = \overline{\varphi}\zeta_i \quad (i=0,\ldots,N).
$$

Then (2.21) is satisfied for each $\overline{\varphi}_i, (i=0,\ldots,N)$. Since each inequality is nonnegative, adding from $i=0$ to N and observing that

$$
\sum_{i=0}^N \nabla\overline{\varphi}_i = \nabla\sum_{i=0}^N \overline{\varphi}_i,
$$

$$
\sum_{i=0}^N \overline{\varphi}_i = \overline{\varphi},
$$

we obtain for any nonnegative function $\overline{\varphi} \in C_0^\infty((0,T) \times \mathbb{R}^n)$

$$\int_0^T \int_\Omega [|\overline{u}_1(t,y) - \overline{u}_2(t,y)|, \operatorname{sgn}(\overline{u}_1(t,y) - \overline{u}_2(t,y)) \\ \times (\overline{f}(t,y,\overline{u}_1(t,y)) - \overline{f}(t,y,\overline{u}_2(t,y)))] \cdot \nabla_{t,y}\overline{\varphi}(t,y)\, J\mathscr{L}(t,y)\, dy\, dt \\ + L \int_0^T \int_\Omega |\overline{u}_1(t,y) - \overline{u}_2(t,y)| \overline{\varphi}(t,y)\, J\mathscr{L}(t,y)\, dy\, dt \\ + \frac{1}{2} \int_0^T \int_{\partial\Omega} \sum_{i,j=1}^{2} (-1)^{i+j} \overline{\Theta}_{ij}(t,y')\overline{\varphi}(t,y')\, J\mathscr{L}_{\mathrm{c}}(t,y')\, d\mathscr{H}^{n-1}(y')\, dt \geqslant 0. \quad (2.22)$$

Returning to the noncylindrical domain Q_T, from (2.22) we get for any $\varphi \in C_0^\infty((0,T) \times \mathbb{R}^n)$, $\varphi \geqslant 0$,

$$\int\int_{Q_T} [|u_1(t,x) - u_2(t,x)|, \operatorname{sgn}(u_1 - u_2)(f(u_1) - f(u_2))] \cdot \nabla_{t,x}\varphi(t,x)\, dx\, dt \\ + L \int\int_{Q_T} |u_1(t,x) - u_2(t,x)|\varphi(t,x)\, dx\, dt \\ + \frac{1}{2} \int_{\Gamma_T} \sum_{i,j=1}^{2} (-1)^{i+j} \Theta_{ij}(r)\varphi(r)\, d\mathscr{H}^n(r) \geqslant 0. \quad (2.23)$$

7. Now we observe that there exists a constant $M > 0$, depending on F and $||u||_\infty$, such that

$$|F(u, u_b) - F(u, v_b)| \leqslant M|u_b - v_b|,$$

hence

$$|F(u_1, u_{b_1}) \cdot \mathrm{n} - F(u_2, u_{b_1}) \cdot \mathrm{n} + F(u_2, u_{b_2}) \cdot \mathrm{n} - F(u_1, u_{b_2}) \cdot \mathrm{n}| \\ \leqslant |F(u_1, u_{b_1}) - F(u_1, u_{b_2})| + |F(u_2, u_{b_2}) - F(u_2, u_{b_1})| \\ \leqslant M|u_{b_1} - u_{b_2}| + M|u_{b_2} - u_{b_1}| \\ = 2M|u_{b_1} - u_{b_2}|.$$

Consequently,

$$\left| \sum_{i,j=1}^{2} (-1)^{i+j} \int_{\Gamma_T} F(u_i(\psi_s(r)), u_{b_j}(r)) \cdot \mathrm{n}(\psi_s(r))\varphi(r)\, d\mathscr{H}^n(r) \right| \\ \leqslant 2M \int_{\Gamma_T} |u_{b_1}(r) - u_{b_2}(r)|\varphi(r)\, d\mathscr{H}^n(r),$$

and taking the limit as $s \to 0^+$, we obtain

$$\left| \sum_{i,j=1}^{2} (-1)^{i+j} \int_{\Gamma_T} \Theta_{ij}(r)\varphi(r)\, d\mathscr{H}^n(r) \right| \leqslant 2M \int_{\Gamma_T} |u_{b_1}(r) - u_{b_2}(r)|\varphi(r)\, d\mathscr{H}^n(r). \tag{2.24}$$

From (2.23) and (2.24), we get for any $\varphi \in C_0^\infty((0,T) \times \mathbb{R}^n)$, $\varphi \geqslant 0$,

$$\begin{aligned} &\int\int_{Q_T} F(u_1(t,x), u_2(t,x)) \cdot \nabla_{t,x}\varphi(t,x)\, dx\, dt \\ &\quad + L \int\int_{Q_T} |u_1(t,x) - u_2(t,x)|\varphi(t,x)\, dx\, dt \\ &\quad + M \int_{\Gamma_T} |u_{b_1}(r) - u_{b_2}(r)|\varphi(r)\, d\mathscr{H}^n(r) \geqslant 0. \end{aligned} \tag{2.25}$$

8. Let $\delta > 0$ be small enough, and $H_\delta(t)$ the Heaviside function. In (2.25), we choose

$$\varphi(t,x) = \phi(t,x)\, (H_\delta * \rho_n)(t),$$

where $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$, $\phi \geqslant 0$, and ρ_n a standard mollifier. Then we have

$$\begin{aligned} &\int_\delta^T \int_{\Omega_t} F(u_1(t,x), u_2(t,x)) \cdot \nabla_{t,x}\phi(t,x)\, dx\, (H_\delta * \rho_n)(t)\, dt \\ &\quad + \int_\delta^T \int_{\Omega_t} |u_1(t,x), u_2(t,x)|\phi(t,x)\, dx\, (H_\delta * {\rho_n}')(t)\, dt \\ &\quad + L \int_\delta^T \int_{\Omega_t} |u_1(t,x) - u_2(t,x)|\phi(t,x)\, dx\, (H_\delta * \rho_n)(t)\, dt \\ &\quad + M \int_\delta^T \int_{\partial\Omega_t} |u_{b_1}(t,r') - u_{b_2}(t,r')|\phi(t,r')\, d\mathscr{H}^{n-1}(r')\, (H_\delta * \rho_n)(t)\, dt \geqslant 0. \end{aligned}$$

Observing that $|H_\delta * \rho_n| \leqslant 1$, and $(H_\delta * \rho'_n)(t)$ converge to the Dirac measure concentrated in δ, we take the limit as $n \to \infty$ and making $\delta \to 0^+$ we get

$$\begin{aligned} &\int\int_{Q_T} F(u_1(t,x), u_2(t,x)) \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \\ &\quad + L \int\int_{Q_T} |u_1(t,x) - u_2(t,x)|\phi(t,x)\, dx\, dt \\ &\quad + M \int_{\Gamma_T} |u_{b_1} - u_{b_2}|\phi(r)\, d\mathscr{H}^n(r) \\ &\quad + \operatorname{ess}\lim_{t \to 0^+} \int_\Omega |u_1(t,x) - u_2(t,x)|\phi(t,x)\, dx \geqslant 0. \end{aligned} \tag{2.26}$$

Moreover, since

$$\left| \int_\Omega |u_1(t,x) - u_2(t,x)| \phi(t,x)\, dx - \int_\Omega |u_{0_1}(x) - u_{0_2}(x)| \phi(0,x)\, dx \right|$$
$$\leqslant C_1 \int_\Omega \{|u_1(t,x) - u_{0_1}(x)| + |u_2(t,x) - u_{0_2}(x)|\}\, dx$$
$$+ C_2 \int_\Omega |\phi(t,x) - \phi(0,x)|\, dx,$$

where C_1 and C_2 are positive constants, utilizing (1.10) we complete the proof of (2.1).

9. Now we finish the proof showing the L^1 contraction. In (2.1), we choose $\phi(t,x) = \zeta(t)$, $\zeta \in C_0^\infty(-\infty, T)$, $\zeta \geqslant 0$ and from Fubini's Theorem, we have

$$\int_0^T \int_{\Omega_t} |u_1(t,x) - u_2(t,x)|\, dx\, \zeta'(t)\, dt$$
$$+ L \int_0^T \int_{\Omega_t} |u_1(t,x) - u_2(t,x)|\, dx\, \zeta(t)\, dt$$
$$+ M \int_0^T \int_{\partial\Omega_t} |u_{b_1}(t,r') - u_{b_2}(t,r')|\, d\mathscr{H}^{n-1}(r')\, \zeta(t)\, dt$$
$$+ \zeta(0) \int_\Omega |u_{0_1}(x) - u_{0_2}(x)|\, dx \geqslant 0. \tag{2.27}$$

Let $\delta > 0$ be small enough and for $t < T$ we take in (2.27) ζ equals the characteristic function on $(\delta, t+\delta)$, i.e., $\zeta = \chi_{(\delta,t+\delta)}$. After mollifying and passing the limit, making $\delta \to 0^+$, we get for almost all $t \in (0,T)$

$$\int_{\Omega_t} |u_1(t,x) - u_2(t,x)|\, dx$$
$$\leqslant \left[\int_\Omega |u_{0_1}(x) - u_{0_2}(x)|\, dx + M \int_0^t \int_{\partial\Omega_\tau} |u_{b_1}(\tau,r') - u_{b_2}(\tau,r')|\, d\mathscr{H}^{n-1}(r')\, d\tau \right]$$
$$+ L \int_0^t \int_{\Omega_\tau} |u_1(t,x) - u_2(t,x)|\, dx\, d\tau.$$

Therefore, from Gronwall's inequality (integral form) we obtain (2.2). $\square$

Remark 2.1. We should make only one change of coordinates in the proof of Theorem 2.1, we made two for clarity. Although, we do not have to work with the jacobian of this one transformation (it is one), the constant L in (2.1) continues depend on the derivatives of the jacobian matrix of this transformation, see (2.15).

3. Existence-uniqueness

The aim of this section is to study the existence of a weak entropy solution to the IBVP (1.1)–(1.3). We prove the existence and uniqueness to (1.1)–(1.3) under the assumptions that the initial-boundary data are L^∞ functions, the flux-function is of class C^1 and Γ_T is smooth.

Here we use the vanishing viscosity method to obtain the desired result, that is, for $\varepsilon > 0$ we study the parabolic perturbation of the IBVP (1.1)–(1.3). We make use of the well-known results of existence, uniqueness and uniform L^∞ bound for quasilinear parabolic problems. Following the vanishing viscosity method, we study the convergence of a family $\{u^\varepsilon\}_{\varepsilon>0}$, solutions to the perturbed problems. An usual procedure in the scalar case, going back to Kruzkov [11], is to derive uniform estimates (with respect to the parameter $\varepsilon > 0$) on

$$||\partial_t u^\varepsilon||_{L^1(Q_T)} \quad \text{and} \quad ||\nabla_x u^\varepsilon||_{L^1(Q_T)}.$$

These estimates and the uniform L^∞ bound on u^ε imply that the family $\{u^\varepsilon\}_{\varepsilon>0}$ is compact in $L^1(Q_T)$. Although, it seems impossible to derive such estimates for noncylindrical domains, even in the one-dimensional case. A problem arises due to the presence of a nonzero time component of the outward unit normal field. In a view of this difficulty, we follow a method introduced by DiPerna [4] and further developed by Szepessy [18] based in the uniqueness of measure-value solutions.

Given $\varepsilon > 0$, we consider the following perturbed problem obtained from (1.1)–(1.3)

$$u_t^\varepsilon + \mathrm{div}_x f(u^\varepsilon) - \varepsilon\, \Delta_x u^\varepsilon = 0 \quad \text{in } Q_T, \tag{3.1}$$

$$u^\varepsilon = u_0^\varepsilon \quad \text{in } \Omega, \tag{3.2}$$

$$u^\varepsilon = u_b^\varepsilon \quad \text{on } \Gamma_T, \tag{3.3}$$

where u_b^ε and u_0^ε are, respectively, boundary and initial smooth data satisfying suitable compatibility conditions on $\partial\Omega$. Standard existence and uniqueness results, see for example Friedman [8], shows that for all $\varepsilon > 0$ problem (3.1)–(3.3) admits a unique solution u^ε satisfying

$$\sup_{\varepsilon>0} ||u^\varepsilon||_{L^\infty} \leqslant C \quad \text{(maximum principle)}.$$

This uniform estimate implies the existence of a subsequence $\{u^{\varepsilon_\ell}\}_{\ell=1}^\infty$ so that

$$u^{\varepsilon_\ell} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q_T).$$

According to Young measures theory, see DiPerna [4] and Tartar [19,20], associated with this subsequence $\{u^{\varepsilon_\ell}\}_{\ell=1}^{\infty}$ there exists a measurable family of Young measures $\nu_{(.)} : Q_T \to \text{Prob}\,(\mathbb{R})$, such that

$$\text{spt}\,\nu_{(t,x)} \subset \{\lambda \,:\, |\lambda| \leqslant C\} \quad \text{for a.e. } (t,x) \in Q_T,$$

where $\text{Prob}\,(\mathbb{R})$ is the space of nonnegative Radon measures over $\mathbb{R}$ with unit mass. For any $g \in C(\mathbb{R})$ the $L^\infty(Q_T)$ weak star limit

$$g \circ u^{\varepsilon_\ell} \overset{*}{\rightharpoonup} \overline{g} \quad \text{in } L^\infty(Q_T)$$

exists, and

$$\overline{g}(t,x) = \int_{\mathbb{R}} g(\lambda)\, d\nu_{(t,x)} =: \langle \nu_{(t,x)}, g(\lambda) \rangle \quad \text{for a.e. } (t,x) \in Q_T.$$

Moreover, u^{ε_ℓ} converges strongly to u in $L^1_{\text{loc}}(Q_T)$ if, and only if, $\nu_{(t,x)}$ reduces to a Dirac measure concentrated at $u(t,x)$, i.e.,

$$\nu_{(t,x)} = \delta_{u(t,x)} \quad \text{for a.e. } (t,x) \in Q_T.$$

Theorem 3.1 (Existence-uniqueness theorem). *Let Q_T be an open smooth subset of $\mathbb{R}^{n+1}$. Let $f \in C^1(\mathbb{R};\mathbb{R}^n)$, $u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty(\Gamma_T;\mathscr{H}^n)$. Then there exists a unique weak entropy solution $u \in L^\infty(Q_T)$ of* (1.1)–(1.3).

Proof. 1. Let $\{\varepsilon_\ell\}_{\ell=1}^{\infty}$ be an arbitrary sequence which converges to zero as $\ell \to \infty$. Let u^{ε_ℓ}, $\ell \in \mathbb{N}$, be the solutions of (3.1) corresponding to smooth uniformly bounded initial-boundary data $u_0^{\varepsilon_\ell}$, $u_b^{\varepsilon_\ell}$, satisfying suitable compatibility conditions on $\partial\Omega$ and

$$u_0^{\varepsilon_\ell} \underset{\ell \to \infty}{\longrightarrow} u_0 \quad \text{in } L^1(\Omega),$$

$$u_b^{\varepsilon_\ell} \underset{\ell \to \infty}{\longrightarrow} u_b \quad \text{in } L^1(\Gamma_T;\mathscr{H}^n).$$

Hence, there exists a function $u \in L^\infty(Q_T)$, such that for a subsequence still denoted u^{ε_ℓ}, we have

$$u^{\varepsilon_\ell} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q_T),$$

and for any $g \in C(\mathbb{R})$

$$g \circ u^{\varepsilon_\ell} \overset{*}{\rightharpoonup} \overline{g} = \langle \nu_{(t,x)}, g(\lambda) \rangle \quad \text{in } L^\infty(Q_T),$$

where $\nu_{(t,x)}$ is the associated family of Young measures.

2. Let $\delta > 0$ be a real number sufficiently small and $s = \min\{\operatorname{dist}((t,x),\Gamma),\ \delta\}$. We define $\tilde{h} : \mathbb{R}^{n+1} \to [-\delta, \delta]$, given by

$$\tilde{h}(t,x) := \begin{cases} s & \text{if } (t,x) \in Q, \\ -s & \text{otherwise.} \end{cases} \tag{3.4}$$

Hence $\tilde{h}(t,x)$ is globally Lipschitz and smooth on the closure of

$$\{(t,x) \in \mathbb{R}^{n+1}; |s| < \delta\}.$$

Further, we define $L := \sup_{0<s<\delta} |\Delta_{t,x} \tilde{h}(t,x)|$. Then the function

$$\xi_{\varepsilon_\ell}(t,x) := 1 - \exp\left(-\frac{\tilde{M} + \varepsilon_\ell L}{\varepsilon_\ell} \tilde{h}(t,x)\right) \tag{3.5}$$

satisfies for any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$, the following weak differential inequality

$$\begin{aligned} \tilde{M} \int\int_{Q_T} |\nabla_{t,x} \xi_{\varepsilon_\ell}| \phi \, dx\, dt - \varepsilon_\ell \int\int_{Q_T} \nabla_x \phi \cdot \nabla_x \xi_{\varepsilon_\ell} \, dx\, dt \\ \leqslant (\tilde{M} + L\varepsilon_\ell) \int_{\Gamma_T} \phi(r) \, d\mathscr{H}^n(r) + \varepsilon_\ell \int\int_{Q_T} \phi_t \, \xi_{\varepsilon_\ell t} \, dx\, dt, \end{aligned} \tag{3.6}$$

where $\tilde{M} > 0$. Moreover, for almost all $(t,x) \in Q_T$ we have

$$\lim_{\ell \to \infty} \int\int_{Q_T} |1 - \xi_{\varepsilon_\ell}| \, dx\, dt = 0, \tag{3.7}$$

$$\lim_{\ell \to \infty} \varepsilon_\ell \int\int_{Q_T} |\nabla_{t,x} \xi_{\varepsilon_\ell}| \, dx\, dt = 0. \tag{3.8}$$

The above result, that is, (3.6)–(3.8) is a suitable modification for noncylindrical domains the similar obtained in Otto [14] for cylindrical ones, see also Málec et al. [13]. As [13,14], we obtain from (3.6) and Eq. (3.1) that u^{ε_ℓ} satisfies for all boundary entropy pairs $\mathscr{F}(u,v) = (\alpha(u,v), \beta(u,v))$, $v \in \mathbb{R}$ and any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$

$$\begin{aligned} &\int\int_{Q_T} \{\mathscr{F}(u^{\varepsilon_\ell}(t,x), v) \cdot \nabla_{t,x} \phi(t,x) + \varepsilon_\ell \alpha(u^{\varepsilon_\ell}(t,x), v) \Delta_x \phi(t,x)\} \xi_{\varepsilon_\ell} \, dx\, dt \\ &\quad + M \int_{\Gamma_T} |u_b^{\varepsilon_\ell}(r) - v| \phi(r) \, d\mathscr{H}^n(r) + \int_\Omega \alpha(u_0^{\varepsilon_\ell}(x), v) \phi(0,x) \xi_{\varepsilon_\ell}(0,x) \, dx \\ &\quad + 2\varepsilon_\ell \int\int_{Q_T} \alpha(u^{\varepsilon_\ell}(t,x), v) \nabla_x \phi(t,x) \cdot \nabla_x \xi_{\varepsilon_\ell}(t,x) \, dx\, dt \end{aligned} \tag{3.9}$$

$$+ \varepsilon_\ell \int\int_{Q_T} (\alpha(u^{\varepsilon_\ell}(t,x), v)\phi(t,x))_t \, (\xi_{\varepsilon_\ell}(t,x))_t \, dx\, dt$$
$$+ L\varepsilon_\ell \int_{\Gamma_T} \alpha(u_b^{\varepsilon_\ell}(r), v)\phi(r)\, d\mathscr{H}^n(r) \geqslant 0.$$

Letting $\ell \to \infty$, we get for any function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$, $\phi \geqslant 0$

$$\begin{aligned} &\int\int_{Q_T} \langle \nu_{(t,x)}, \mathscr{F}(\lambda, v) \rangle \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \\ &+ M \int_{\Gamma_T} |u_b(r) - v| \phi(r)\, d\mathscr{H}^n(r) \\ &+ \int_\Omega \alpha(u_0(x), v)\phi(0,x)\, dx \geqslant 0, \end{aligned} \tag{3.10}$$

where we have used (3.7)–(3.8).

3. Suppose that $\nu^1_{(t,x)}$ and $\nu^2_{(t,x)}$ are two associated Young measures satisfying (3.10), corresponding to two subsequences of $\{u^{\varepsilon_\ell}\}$ with the $L^\infty(Q_T)$ weak star limit u_1 and u_2 respectively. In (3.10), we choose $\phi \in C_0^\infty(Q_T)$, $\phi \geqslant 0$ and $\mathscr{F}$ exactly the Kružkov's entropies, then for each $\nu^i_{(t,x)}$, $(i = 1, 2)$, we obtain

$$\int\int_{Q_T} \langle \nu^i_{(t,x)}, F(\lambda_1, \lambda_2) \rangle \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \geqslant 0. \tag{3.11}$$

Formally, we have for any nonnegative function $\phi \in C_0^\infty(Q_T)$

$$\begin{aligned} &\int\int_{Q_T} \langle \nu^1_{(t,x)} \otimes \nu^2_{(t,x)}, F(\lambda_1, \lambda_2) \rangle \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \\ &= -\int\int_{Q_T} \phi(t,x) \int_{\mathbb{R}} \mathrm{div}_{t,x} \langle \nu^1_{(t,x)}, F \rangle \, d\nu^2_{(t,x)}\, dx\, dt \\ &\quad - \int\int_{Q_T} \phi(t,x) \int_{\mathbb{R}} \mathrm{div}_{t,x} \langle \nu^2_{(t,x)}, F \rangle \, d\nu^1_{(t,x)}\, dx\, dt \\ &= \int_{\mathbb{R}} \int\int_{Q_T} \langle \nu^1_{(t,x)}, F \rangle \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt\, d\nu^2_{(t,x)} \\ &\quad + \int_{\mathbb{R}} \int\int_{Q_T} \langle \nu^2_{(t,x)}, F \rangle \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt\, d\nu^1_{(t,x)} \geqslant 0, \end{aligned} \tag{3.12}$$

where we have used (3.11), the Fubini's Theorem and the product rule. This argument can be made rigorous by a standard mollification (see Szepessy [18]).

4. Now, we claim that $\nu^1_{(t,x)}$ and $\nu^2_{(t,x)}$ have supports consisting of a common single point $u(t,x)$ for a.e. $(t,x) \in Q_T$, i.e.,

$$\nu^1_{(t,x)} = \nu^2_{(t,x)} = \delta_{u(t,x)}.$$

Indeed, in (3.12) choosing $\phi(t,x)=\zeta(t)$, $\zeta \in C_0^\infty(0,T)$, $\zeta \geqslant 0$, we have

$$\int_0^T \zeta'(t)\, A(t)\, dt \geqslant 0,$$

where

$$A(t) := \int_{\Omega_t} \langle \nu^1_{(t,x)} \otimes \nu^2_{(t,x)}, |\lambda_1 - \lambda_2| \rangle\, dx.$$

Letting ζ tends to the characteristic function on (s,t), $0<s\leqslant t$, we get

$$A(s) - A(t) \geqslant 0. \tag{3.13}$$

Now,

$$\begin{aligned} A(t) &\leqslant \int_{\Omega_t} \langle \nu^1_{(t,x)} \otimes \nu^2_{(t,x)}, |\lambda_1 - u_0| + |\lambda_2 - u_0| \rangle\, dx \\ &= \int_{\Omega_t} \langle \nu^1_{(t,x)}, |\lambda_1 - u_0| \rangle\, dx + \int_{\Omega_t} \langle \nu^2_{(t,x)}, |\lambda_2 - u_0| \rangle\, dx \end{aligned}$$

and the initial data is assumed in the strong sense, so that both terms in the right hand side of the above inequality converges to zero as $t \to 0^+$. Then, making $s \to 0^+$ in (3.13), we get

$$A(t) = \int_{\Omega_t} \langle \nu^1_{(t,x)} \otimes \nu^2_{(t,x)}, |\lambda_1 - \lambda_2| \rangle\, dx \equiv 0.$$

Consequently, for almost everywhere $(t,x) \in Q_T$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\lambda_1 - \lambda_2|\, d\nu^1_{(t,x)}\, d\nu^2_{(t,x)} = 0. \tag{3.14}$$

Thus, the assertion follows by (3.14) and a contradiction argument. Indeed, following Szepessy [18] we suppose that there exists $w_1 \neq w_2$ with

$$w_1 \in \operatorname{spt} \nu^1_{(t,x)}, \quad w_2 \in \operatorname{spt} \nu^2_{(t,x)}$$

for each $(t,x) \in Q_T$ and such that (3.14) holds. Then, by definition there are

$$0 \leqslant \phi_j \in C_0(\mathbb{R}), \quad w_j \in \operatorname{spt} \phi_j \quad (j=1,2)$$

such that $\operatorname{spt} \phi_1 \cap \operatorname{spt} \phi_2 = \emptyset$ and

$$\langle \nu^1_{(t,x)}, \phi_1 \rangle > 0, \quad \langle \nu^2_{(t,x)}, \phi_2 \rangle > 0.$$

Thus by Fubini's theorem and (3.14)

$$0 < \int\int \phi_1 \phi_2\, d\nu^1_{(t,x)}\, d\nu^2_{(t,x)} \leqslant \left\| \frac{\phi_1 \phi_2}{w_2 - w_1} \right\|_{L^\infty} \int\int |w_2 - w_1|\, d\nu^1_{(t,x)}\, d\nu^2_{(t,x)} = 0,$$

which is a contradiction.

5. Now in (3.10) with $v_{(t,x)} = \delta_{u(t,x)}$, we obtain that u is the unique function that satisfies

$$\begin{aligned} \int \int_{Q_T} & \mathscr{F}(u(t,x), v) \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \\ & + M \int_{\Gamma_T} |u_b(r) - v|\phi(r)\, d\mathscr{H}^n(r) \\ & + \int_{\Omega} \alpha(u_0(x), v)\phi(0,x)\, dx \geqslant 0, \end{aligned}$$

for all boundary entropy pairs $\mathscr{F}(u,v) = (\alpha(u,v), \beta(u,v))$, $v \in \mathbb{R}$ and any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$. Consequently, by Theorem 1.1 $u \in L^\infty(Q_T)$ satisfies (1.9)–(1.11). $\square$

4. Passage to Lipschitz boundary

In this section we show how we can extend the last result, i.e. Theorem 3.1 to regularizable Lipschitz domains. In order to obtain this result we utilize the notion of regularized problems, that is, for $\delta > 0$ we make a regularization of the IBVP (1.1)–(1.3). To study the convergence of a family $\{u^\delta\}_{\delta>0}$, solutions to the regularized problems, we utilize the same techniques of the Section 3. The stability result does not pass to regularizable Lipschitz domains, due that, we do not have a uniform bound (if respect to the parameter $\delta > 0$) on L in (2.1).

In Section 1 we have defined Q_T from the cylinder $(0,T) \times \Omega$ by a bi-Lipschitz map $\mathscr{L}(t,y)$, hence the lateral boundary Γ_T was a Lipschitz variety and since we have assumed that $\partial\Omega$ is a regular deformable Lipschitz boundary, Γ_T possess the same regularity. Now without loss of generality, we can assume that the map $\mathscr{L}$ is defined in a domain of $\mathbb{R}^{n+1}$ containing the set $\mathbb{R} \times \Omega$ and its deformations. Analogously, we do not continue considering $x(t,y) = y$ for $t \leqslant 0$ and assume that $\partial\Omega$ is smooth. By abuse of notation we continue denoting by Ω the set $\overline{Q}_T \cap \{t = 0\}$.

Definition 4.1. Let E be an open set of $\mathbb{R}^n$ with smooth boundary ∂E. Let $L : S \to \mathbb{R}^n$ be a bi-Lipschitz map over its image, where the set $S \subset \mathbb{R}^n$ contain E and its deformations. Set $U := L(E)$ and its boundary $\partial U := L(\partial E)$. We say that $(L, \partial U)$ is a regularizable Lipschitz boundary pair provided that the following hold:

(i) There exists a family $\{L^\delta\}_{\delta>0}$, such that for each $\delta > 0$, L^δ is a diffeomorphism of class C^∞ over its image, $L^\delta \to L$ uniformly, $DL^\delta \to DL$ $\mathscr{L}^n$-a.e. and $\sup_\delta ||DL^\delta||_{L^\infty} < \infty$. In this case, we define

$$U^\delta := L^\delta(E), \quad \partial U^\delta := L^\delta(\partial E)$$

and $\Psi^\delta := L^\delta \circ L^{-1}$, which converges uniformly to the identity map;

(ii) The restriction of Ψ^δ over ∂U, denoted Ψ_c^δ, must be a homeomorphism bi-Lipschitz over its image and

$$\lim_{\delta \to 0^+} D(\Psi_c^\delta \circ \tilde{\gamma}) = D\tilde{\gamma} \quad \text{in } L^1_{\text{loc}}(B),$$

where $\tilde{\gamma}(y) = (y, \gamma(y))$ with $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ given by the Lipschitz boundary definition, see Evans and Gariepy [6], and B is the greatest open set such that $\tilde{\gamma}(B) \subset \partial U$.

In this context, we usually say that U is a regularizable Lipschitz domain.

We assume that $(\mathscr{L}, \Gamma)$ is a regularizable Lipschitz boundary pair. So there exists a family $\{\mathscr{L}^\delta\}_{\delta>0}$ as in Definition 4.1 and we define: $\Psi^\delta \coloneqq \mathscr{L}^\delta \circ \mathscr{L}^{-1}$; $\Psi_c^\delta \coloneqq \Psi^\delta|_\Gamma$. Moreover,

$$Q^\delta \coloneqq \mathscr{L}^\delta(\mathbb{R} \times \Omega), \quad \Gamma^\delta \coloneqq \mathscr{L}^\delta(\mathbb{R} \times \partial\Omega),$$

and for some $T \in \mathbb{R}$, $T > 0$

$$Q_T^\delta \coloneqq \mathscr{L}^\delta((0, T) \times \Omega), \quad \Gamma_T^\delta \coloneqq \mathscr{L}^\delta((0, T) \times \partial\Omega).$$

Considering $\{Q_T^\delta\}$ and $\{\Gamma_T^\delta\}$, we establish from (1.1)–(1.3) the following family of regularized problems:

For each $\delta > 0$, find u^δ satisfying

$$u_t^\delta + \operatorname{div}_x f(u^\delta) = 0 \quad \text{in } Q_T^\delta, \tag{4.1}$$

$$u^\delta = u_0^\delta \quad \text{in } \Omega^\delta, \tag{4.2}$$

$$u^\delta = u_b^\delta \quad \text{on } \Gamma_T^\delta, \tag{4.3}$$

where $\Omega^\delta = \overline{Q^\delta}_T \cap \{t = 0\}$ and u_0^δ, u_b^δ are defined in the following form. Let $\bar{u}_0 \in C^0(\mathbb{R}^n)$, $\bar{u}_b \in C^0(\mathbb{R}^{n+1})$ be the extensions of u_0 and u_b by the Lusin Theorem, see Evans and Gariepy [6], such that

$$\mathscr{L}^n(\{x \in \Omega : \bar{u}_0(x) \neq u_0(x)\}) = 0,$$

$$\mathscr{H}^n(\{r \in \Gamma_T : \bar{u}_b(r) \neq u_b(r)\}) = 0.$$

Moreover, satisfying

$$||\bar{u}_0||_\infty \leqslant ||u_0||_\infty \quad \text{and} \quad ||\bar{u}_b||_\infty \leqslant ||u_b||_\infty.$$

Then, we define

$$u_0^\delta \coloneqq \bar{u}_0|_{\Omega^\delta},$$

$$u_b^\delta \coloneqq \bar{u}_b|_{\Gamma_T^\delta}.$$

Theorem 4.1. *Let Q_T be an open regularizable Lipschitz subset of $\mathbb{R}^{n+1}$. Let $f \in C^1(\mathbb{R};\mathbb{R}^n)$, $u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty(\Gamma_T;\mathscr{H}^n)$. Then there exists a unique weak entropy solution $u \in L^\infty(Q_T)$ of* (1.1)–(1.3).

Proof. Let $\{\delta_\ell\}_{\ell=1}^\infty$ be an arbitrary sequence which converges to zero as $\ell \to \infty$. Let u^{δ_ℓ}, $\ell \in \mathbb{N}$, be the solutions of (4.1)–(4.3) by Theorem 3.1 with $u_0^\delta = u_0^{\delta_\ell}$ and $u_b^\delta = u_b^{\delta_\ell}$. Since $u_0^{\delta_\ell}$ and $u_b^{\delta_\ell}$ are uniformly bounded, by Corollary 1.1 we have

$$\sup_\ell ||u^{\delta_\ell}||_{L^\infty} \leqslant C.$$

Hence, there exists a function $u \in L^\infty(Q_T)$, such that for a subsequence still denoted u^{δ_ℓ}, we have

$$u^{\delta_\ell} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(Q_T),$$

and for any $g \in C(\mathbb{R})$

$$g \circ u^{\delta_\ell} \overset{*}{\rightharpoonup} \overline{g} = \langle \nu_{(t,x)}, g(\lambda) \rangle \quad \text{in } L^\infty(Q_T).$$

Now, we are going to show that u satisfies (1.1)–(1.3). By Theorem 1.1, we have that for all boundary entropy pairs $\mathscr{F}(u,v) = (\alpha(u,v), \beta(u,v))$, $v \in \mathbb{R}$, there exists a constant $M > 0$ independent of δ_ℓ, such that for any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^n)$ and for each $\ell \in \mathbb{N}$, u^{δ_ℓ} satisfies

$$\begin{aligned} \int\int_{Q_T^{\delta_\ell}} & \mathscr{F}(u^{\delta_\ell}(t,x), v)) \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \\ & + M \int_{\Gamma_T^{\delta_\ell}} |u_b^{\delta_\ell}(r) - v| \phi(r)\, d\mathscr{H}^n(r) \\ & + \int_{\Omega^{\delta_\ell}} \alpha(u_0^{\delta_\ell}(x), v)\phi(0,x)\, dx \geqslant 0. \end{aligned} \tag{4.4}$$

Letting $\ell \to \infty$, we get

$$\begin{aligned} \int\int_{Q_T} & \langle \nu_{(t,x)}, \mathscr{F}(\lambda, v) \rangle \cdot \nabla_{t,x}\phi(t,x)\, dx\, dt \\ & + M \int_{\Gamma_T} |u_b(r) - v| \phi(r)\, d\mathscr{H}^n(r) \\ & + \int_{\Omega} \alpha(u_0(x), v)\phi(0,x)\, dx \geqslant 0. \end{aligned} \tag{4.5}$$

Indeed, we observe the following terms:

$$\begin{aligned}
&\left|\int\int_{Q_T} \langle \nu_{(t,x)}, \mathscr{F}(\lambda, v)\rangle \cdot \nabla\phi - \int\int_{Q_T^{\delta_\ell}} \mathscr{F}(u^{\delta_\ell}, v)\cdot\nabla\phi\right| \\
&\quad = \left|\int\int_{Q_T} \langle \nu_{(t,x)}, \mathscr{F}(\lambda, v)\rangle \cdot \nabla\phi - \int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}(\Psi^{\delta_\ell}), v)\cdot\nabla\phi^{\delta_\ell} J\Psi^{\delta_\ell}\right| \\
&\quad \leqslant \left|\int\int_{Q_T} \langle \nu_{(t,x)}, \mathscr{F}(\lambda, v)\rangle \cdot \nabla\phi - \int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}, v)\cdot\nabla\phi\right| \\
&\qquad + \left|\int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}, v)\cdot\nabla\phi - \int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}(\Psi^{\delta_\ell}), v)\cdot\nabla\phi\right| \\
&\qquad + \left|\int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}(\Psi^{\delta_\ell}), v)\cdot\nabla\phi - \int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}(\Psi^{\delta_\ell}), v)\cdot\nabla\phi^{\delta_\ell}\right| \\
&\qquad + \left|\int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}(\Psi^{\delta_\ell}), v)\cdot\nabla\phi^{\delta_\ell} - \int\int_{Q_T} \mathscr{F}(u^{\delta_\ell}(\Psi^{\delta_\ell}), v)\cdot\nabla\phi^{\delta_\ell} J\Psi^{\delta_\ell}\right| \xrightarrow[\ell\to\infty]{} 0,
\end{aligned}$$

where $\nabla\phi^{\delta_\ell} := [D\Psi^{\delta_\ell}]^T\nabla\phi(\Psi^{\delta_\ell})$. Moreover, we have used the uniform bound of u^{δ_ℓ}, the properties of Ψ^{δ_ℓ}, the Dominated Convergence Theorem and the Change of variables formula [6,7]. Analogously, we have

$$\begin{aligned}
&\left|\int_\Omega \alpha(u_0, v)\phi(0)\,dx - \int_{\Omega^{\delta_\ell}} \alpha(u_0^{\delta_\ell}, v)\phi(0)\,dx\right| \\
&\quad = \left|\int_\Omega \alpha(\overline{u}_0, v)\phi(0)\,dx - \int_{\Omega^{\delta_\ell}} \alpha(\overline{u}_0, v)\phi(0)\,dx\right| \\
&\quad = \left|\int_\Omega \alpha(\overline{u}_0, v)\phi(0)\,dx - \int_{\Omega} \alpha(\overline{u}_0(\Psi^{\delta_\ell}(0)), v)\phi(\Psi^{\delta_\ell}(0))J\Psi^{\delta_\ell}(0)\,dx\right| \\
&\quad \leqslant C\left[\int_\Omega |Id - \Psi^{\delta_\ell}|\,dx + \int_\Omega |1 - J\Psi^{\delta_\ell}|\right] \xrightarrow[\ell\to\infty]{} 0,
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{\Gamma_T} |u_b(r) - v|\phi(r)\,d\mathscr{H}^n(r) - \int_{\Gamma_T^{\delta_\ell}} |u_b^{\delta_\ell}(r) - v|\phi(r)\,d\mathscr{H}^n(r)\right| \\
&\quad = \left|\int_{\Gamma_T} |\overline{u}_b(r) - v|\phi(r)\,d\mathscr{H}^n(r) - \int_{\Gamma_T^{\delta_\ell}} |\overline{u}_b(r) - v|\phi(r)\,d\mathscr{H}^n(r)\right| \\
&\quad = \left|\int_{\Gamma_T} |\overline{u}_b(r) - v|\phi(r)\,d\mathscr{H}^n(r) - \int_{\Gamma_T} |\overline{u}_b(\Psi_c^{\delta_\ell}(r)) - v|\phi(\Psi_c^{\delta_\ell}(r))J\Psi_c^{\delta_\ell}\,d\mathscr{H}^n(r)\right| \\
&\quad \leqslant C\left[\int_{\Gamma_T} |Id - \Psi_c^{\delta_\ell}|\,d\mathscr{H}^n + \int_{\Gamma_T} |1 - J\Psi_c^{\delta_\ell}|\,d\mathscr{H}^n\right] \xrightarrow[\ell\to\infty]{} 0.
\end{aligned}$$

Then we showed (4.5) and repeating procedures (3)–(4) in the proof of Theorem 3.1, we obtain that $\nu_{(t,x)} \equiv \delta_{u(t,x)}$ for a.e. $(t,x)\in Q_T$. Consequently, $u\in L^\infty(Q_T)$ is the unique function which satisfies

$$\begin{aligned}
&\int\int_{Q_T} \mathscr{F}(u(t,x),v)\cdot\nabla_{t,x}\phi(t,x)\,dx\,dt \\
&+ M\int_{\Gamma_T} |u_b(r)-v|\phi(r)\,d\mathscr{H}^n(r) \\
&+ \int_{\Omega}\alpha(u_0(x),v)\phi(0,x)\,dx \geqslant 0,
\end{aligned}$$

for all boundary entropy pair $\mathscr{F}(u,v)$ and any nonnegative function $\phi\in C_0^\infty((-\infty,T)\times\mathbb{R}^n)$. By Theorem 1.1 and Definition 1.3 u is the weak entropy solution of (1.1)–(1.3). $\square$

Appendix A. An example of regularizable Lipschitz domain

Let a,b be real positive numbers and set

$$\Omega = \{(y_1,y_2)\in\mathbb{R}^2 / -\frac{b}{a}\sqrt{a^2-y_1^2} < y_2 < \frac{b}{a}\sqrt{a^2-y_1^2}, -a<y_1<a\}.$$

Consider a real positive Lipschitz function $\gamma(t,y_1)>0$, and define

$$\mathscr{L}(t,y_1,y_2) := \begin{cases} t=t,\\ x_1=y_1,\\ x_2=y_2\ \gamma(t,y_1).\end{cases}$$

Moreover, we consider

$$\begin{aligned}
Q_T &= \mathscr{L}((0,T)\times\Omega)\\
&= \{(t,x_1,x_2)\in\mathbb{R}^3; -\frac{b}{a}\gamma(t,x_1)\sqrt{a^2-x_1^2} < x_2 < \frac{b}{a}\gamma(t,x_1)\sqrt{a^2-x_1^2},\\
&\quad -a<x_1<a, 0<t<T\}.
\end{aligned}$$

It is immediate that $\mathscr{L}$ is a bi-Lipschitz function, in fact, $\mathscr{L}^{-1}(t,x_1,x_2) = (t,x_1,x_2/\gamma(t,x_1))$.

Now, let $\{\gamma^\delta\}_{\delta>0}$ be a family of C^∞ functions such that

$$\begin{cases} \gamma^\delta \geqslant \gamma,\\ \gamma^\delta \to \gamma \text{ uniformly},\\ D\gamma^\delta \to D\gamma \text{ a.e.},\\ \sup_\delta ||D\gamma^\delta||_{L^\infty} < \infty,\end{cases}$$

and we naturally define the family $\{\mathscr{L}^\delta\}_{\delta>0}$ by

$$\mathscr{L}^\delta(t, y_1, y_2) := \begin{cases} t = t, \\ x_1 = y_1, \\ x_2 = y_2\ \gamma^\delta(t, y_1). \end{cases}$$

Then, we have $\Psi^\delta(t, x_1, x_2) = (t, x_1, x_2\ \gamma^\delta(t, x_1)/\gamma(t, x_1))$.

The set Q_T as defined above is an example of a regularizable Lipschitz domain in $\mathbb{R}^3$. In fact, we have a family of examples, since we can change a, b and principally γ.

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